Solutions to Exercises

For many of the exercises, drawing a diagram will be found extremely helpful.

Chapter 1

1.1 (i) *The triangle inequality.*

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$. Then

$$\begin{aligned}
|x+y|^2 &= \sum_{i=1}^{n}(x_i + y_i)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2 \\
&\leq \sum_{i=1}^{n} x_i^2 + 2\left(\sum_{i=1}^{n} x_i^2\right)^{1/2}\left(\sum_{i=1}^{n} y_i^2\right)^{1/2} + \sum_{i=1}^{n} y_i^2 \\
&= \left(\left(\sum_{i=1}^{n} x_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} y_i^2\right)^{1/2}\right)^2 = (|x| + |y|)^2
\end{aligned}$$

where we have used Cauchy's inequality.

(ii) *The reverse triangle inequality.*

Write $y = z - x$ so $x = z - y$. Then (i) becomes $|z| \leq |z-y| + |y|$ or $|z| - |y| \leq |z-y|$. Interchanging the roles of y and z we also have $|y| - |z| \leq |y-z| = |z-y|$. Thus $||z| - |y|| = \max\{|z|-|y|, |y|-|z|\} \leq |z-y|$, which is the desired inequality.

(iii) *Triangle inequality - metric form.*

We have

$$|x-y| = |(x-z) + (z-y)| \leq |x-z| + |z-y|$$

using triangle inequality (i).

1.2 Let $x \in A_{\delta+\delta'}$. Then there exists $a \in A$ such that $|x-a| \le \delta+\delta'$. If $x = a$, then clearly $x \in (A_\delta)_{\delta'}$. Otherwise let y be the point on the line segment $[a, x]$ distance δ from a. Thus $y = a + \delta(x-a)/|x-a|$, so $|y-a| = \delta|x-a|/|x-a| = \delta$, so $y \in A_\delta$. Moreover, $x - y = x - a - \delta(x-a)/|x-a| = (x-a)[1-\delta/|x-a|]$, so $|x-y| = |x-a| - \delta \le \delta + \delta' - \delta = \delta'$. As $y \in A_\delta, x \in (A_\delta)_{\delta'}$, so $A_{\delta+\delta'} \subseteq (A_\delta)_{\delta'}$.

Now let $x \in (A_\delta)_{\delta'}$. We may find $y \in A_\delta$ such that $|x-y| \le \delta'$, and then we may find $a \in A$ such that $|y-a| \le \delta$. By the triangle inequality, Exercise 1.1(iii), $|x-a| \le |x-y| + |y-a| \le \delta' + \delta$, so $x \in A_{\delta+\delta'}$. Thus $(A_\delta)_{\delta'} \subseteq A_{\delta+\delta'}$. We conclude $(A_\delta)_{\delta'} = A_{\delta+\delta'}$.

1.3 Let A be bounded, that is A has finite diameter, so $\sup_{x,y\in A} |x-y| = d < \infty$, where d is the diameter of A. Let a be any point of A. Then for all $x \in A$, $|x-a| \le d$, so that $|x| = |a + (x-a)| \le |a| + |x-a| \le |a| + d$, using the triangle inequality, Exercise 1.1(i). Thus, setting $r = |a| + d$, we have $x \in B(0, r)$. We conclude $A \subseteq B(0, r)$.

If $A \subseteq B(0, r)$ and $x, y \in A$, then $|x-y| \le |x| + |y| \le r + r = 2r$, so $\operatorname{diam} A \le 2r$, and in particular A is of finite diameter.

1.4 (i) A non-empty finite set is closed but not open, with $\overline{A} = A$, and $\operatorname{int}A = \emptyset$.

(ii) The interval $(0, 1)$ is open but not closed, with $\overline{(0,1)} = [0, 1]$ and $\operatorname{int}(0, 1) = (0, 1)$.

(iii) The interval $[0, 1]$ is closed but not open, with $\overline{[0,1]} = [0, 1]$ and $\operatorname{int}[0, 1] = (0, 1)$.

(iv) The half-open interval $[0, 1)$ is neither open or closed, with $\overline{[0,1)} = [0, 1]$ and $\operatorname{int}[0, 1) = (0, 1)$.

(v) The set $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$ is closed but not open, with $\overline{A} = A$ and $\operatorname{int}A = \emptyset$.

1.5 Following the usual construction, the middle third Cantor set may be written $F = \bigcap_{k=0}^{\infty} E_k$, where E_k consists of the union of 2^k disjoint closed intervals in $[0, 1]$, each of length 3^{-k}. For each k, E_k is closed since it is the union of finitely many closed sets. Since the intersection of any collection of closed sets is closed (see Exercise 1.6), we conclude that F is closed. F is a subset of $[0, 1]$ so it is bounded, and hence F is compact.

To show that F is totally disconnected, suppose $x, y \in F$ with $x < y$. Then we can find an E_k such that x and y belong to different intervals $[a, b]$ and $[c, d]$ of E_k with $b < c$. Let $b < p < c$. Then F is contained in the union of the disjoint open intervals $(-1, p)$ and $(p, 2)$, with $x \in (-1, p)$ and $y \in (p, 2)$. Thus F is totally disconnected.

Since F is closed, $\overline{F} = F$. Since F contains no open interval, $\text{int}F = \emptyset$, and thus $\partial F = \overline{F} \backslash \text{int}F = F$.

1.6 Let $\{A_i : i \in I\}$ be a collection of open subsets of $\mathbb{R}^n$ and let $A = \bigcup_{i \in I} A_i$. If $x \in A$, then x belongs to one of the sets, A_j, say. Since A_j is open, there exists $r > 0$ such that $B(x, r) \subset A_j \subset A$, and hence A is open.

Now let $\{A_1, A_2, \ldots, A_k\}$ be a finite collection of open subsets of $\mathbb{R}^n$ and let $A = \bigcap_{i=1}^k A_i$. If $x \in A$, then x belongs to each of the open sets A_i and hence, for each $i = 1, \ldots, k$, there exists $r_i > 0$ such that $B(x, r_i) \subset A_i$. Letting $r = \min_{1 \le i \le k} r_i > 0$, then $B(x, r) \subset B(x, r_i) \subset A_i$ for all i, so that $B(x, r) \subset A$ and hence A is open.

Let $A \subset \mathbb{R}^n$ and let $B = \mathbb{R}^n \backslash A$ be the complement of A. First assume that B is not open. Then there exists $x \in B$ such that, for every positive integer k, the ball $B(x, 1/k)$ is not contained in B and we may choose a sequence $x_k \in B(x, 1/k) \setminus B$, so $x_k \in A$ and $x_k \to x \notin A$, so A is not closed. Thus if A is closed then B must be open.

Now suppose that A is not closed so that there exists a sequence of points $x_k \in A$ with $x_k \to x \in B = \mathbb{R}^n \backslash A$. It follows that, for every $r > 0$, there is some $x_k \in B(x, r) \backslash B$ so that $B(x, r) \not\subset B$, giving that B is not open. Thus if B is open then $A = \mathbb{R}^n \backslash B$ must be closed.

Now let $\{B_i : i \in I\}$ be a collection of closed subsets of $\mathbb{R}^n$ and let $B = \bigcap_{i \in I} B_i$. Each of the sets $A_i = \mathbb{R}^n \backslash B_i$ is open. Thus

$$A = \bigcup_{i \in I} A_i = \bigcup_{i \in I} (\mathbb{R}^n \backslash B_i) = \mathbb{R}^n \backslash B$$

is open and hence B is closed.

Similarly, if $\{B_i : i = 1, \ldots, k\}$ is a finite collection of closed subsets of $\mathbb{R}^n$ and $B = \bigcup_{i=1}^k B_i$, then each of the sets $A_i = \mathbb{R}^n \backslash B_i$ is open and hence

$$A = \bigcap_{i=1}^{k} A_i = \bigcap_{i=1}^{k} (\mathbb{R}^n \backslash B_i) = \mathbb{R}^n \backslash B$$

is open so that B is closed.

1.7 Recall that a subset of $\mathbb{R}^n$ is compact if and only if it is both closed and bounded. Exercise 1.6 showed that the intersection of any collection of closed subsets of $\mathbb{R}^n$ is closed. Thus, if $A_1 \supset A_2 \supset \cdots$ is a decreasing sequence of non-empty compact subsets of $\mathbb{R}^n$ then $A = \bigcap_{k=1}^\infty A_k$ is certainly closed. It is also bounded, since it is a subset of A_1 which is bounded, so A is compact.

To show that A is non-empty we argue by contradiction. Suppose that $\bigcap_{k=1}^{\infty} A_k = \emptyset$ so that $\mathbb{R}^n = \mathbb{R}^n \backslash \bigcap_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty}(\mathbb{R}^n \backslash A_k)$. Then $A_1 \subset \bigcup_{k=1}^{\infty}(\mathbb{R}^n \backslash A_k)$. Since A_1 is compact, it follows that A_1 is contained in the union of finitely many of the open sets $\mathbb{R}^n \backslash A_k$. Since $\mathbb{R}^n \backslash A_1 \subset \mathbb{R}^n \backslash A_2 \subset \cdots$, it follows that $A_1 \subset (\mathbb{R}^n \backslash A_k)$ for some k. This is impossible, since $A_k \subset A_1$ and $A_k \neq \emptyset$, for each k.

1.8 The half-open interval $[0, 1)$ is a Borel subset of $\mathbb{R}$ since, for example,

$$[0, 1) = [0, 2] \cap (-1, 1).$$

where $[0, 2]$ is closed and hence a Borel set and $(-1, 1)$ is open and hence a Borel set.

1.9 Let A_k be the set of numbers in $[0, 1]$ whose kth digit is 5. Then A_k is union of 10^{k-1} half open intervals, so is Borel. Then

$$F = \bigcap_{j=1}^{\infty} \bigcup_{k=j}^{\infty} A_k,$$

as $x \in F$ if and only if $x \in A_k$ for arbitrarily large k. Thus F is formed as the countable intersection of a countable union of Borel sets, so is Borel.

1.10 Let $x = (x_1, x_2),\ y = (y_1, y_2),\ a = (a_1, a_2)$. We may write the transformation S as

$$S(x_1, x_2) = (cx_1 \cos\theta - cx_2 \sin\theta + a_1,\ cx_1 \sin\theta + cx_2 \cos\theta + a_2)$$

so

$$\begin{aligned}
&|S(x_1, x_2) - S(y_1, y_2)|^2 \\
&= c^2|((x_1 - y_1)\cos\theta - (x_2 - y_2)\sin\theta, (x_1 - y_1)\sin\theta + (x_2 - y_2)\cos\theta)|^2 \\
&= c^2((x_1 - y_1)^2\cos^2\theta + (x_2 - y_2)^2)\sin^2\theta - 2(x_1 - y_1)(x_2 - y_2)\sin\theta\cos\theta \\
&+ c^2((x_1 - y_1)^2\sin^2\theta + (x_2 - y_2)^2)\cos^2\theta + 2(x_1 - y_1)(x_2 - y_2)\sin\theta\cos\theta \\
&= c^2((x_1 - y_1)^2 + (x_2 - y_2)^2) = c^2|(x_1, x_2) - (y_1, y_2)|^2,
\end{aligned}$$

using that $\cos^2\theta + \sin^2\theta = 1$. Thus $|S(x) - S(y)| = c|x - y|$, so S is a similarity of ratio c.

Note that $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ gives the vector $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ rotated about the origin by an anticlockwise angle θ. Thus the geometrical effect of the similarity S is a dilation about the origin of scale c, followed by a rotation through angle θ, followed by a translation by the vector $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$.

1.11 (i) Since $\sin x \to \sin 0 = 0$ as $x \to 0$, we have

$$\underline{\lim}_{x\to 0} \sin x = \overline{\lim}_{x\to 0} \sin x = \lim_{x\to 0} \sin x = 0.$$

(ii) We know that

$$-1 \le \sin(1/x) \le 1, \quad \text{for } x > 0$$

so that

$$-1 \le \underline{\lim}_{x\to 0} \sin(1/x) \le \overline{\lim}_{x\to 0} \sin(1/x) \le 1.$$

Moreover, for each $n = 1, 2, \dots,$

$$\sin(1/x_n) = -1, \quad \text{for } x_n = 1/(2n + 3/2)\pi \to 0$$

and

$$\sin(1/y_n) = 1, \quad \text{for } y_n = 1/(2n + 1/2)\pi.$$

Thus

$$\underline{\lim}_{x\to 0} \sin(1/x) \le -1 \text{ and } \overline{\lim}_{x\to 0} \sin(1/x) \ge 1,$$

so $\underline{\lim}_{x\to 0} \sin(1/x) = -1$ and $\overline{\lim}_{x\to 0} \sin(1/x) = 1$.

(iii) We have $|x^2 + x\sin(1/x)| \le |x^2| + |x| \to 0$ as $x \to 0$. Thus

$$\begin{aligned}\underline{\lim}_{x\to 0}(x^2 + (3 + x)\sin(1/x)) &= \underline{\lim}_{x\to 0}(x^2 + x\sin(1/x)) \\ &\quad + \underline{\lim}_{x\to 0} 3\sin(1/x) \\ &= 0 - 3 = -3\end{aligned}$$

using part (ii). Similarly

$$\begin{aligned}\overline{\lim}_{x\to 0}(x^2 + (3 + x)\sin(1/x)) &= \overline{\lim}_{x\to 0}(x^2 + x\sin(1/x)) \\ &\quad + \overline{\lim}_{x\to 0} 3\sin(1/x) \\ &= 0 + 3 = 3.\end{aligned}$$

1.12 If $f, g : [0, 1] \to \mathbb{R}$ are Lipschitz functions, then there exist $c_1, c_2 > 0$ such that

$$|f(x) - f(y)| \le c_1|x - y| \text{ and } |g(x) - g(y)| \le c_2|x - y| \quad (x, y \in [0, 1]).$$

It follows that

$$\begin{aligned}|f(x)+g(x)-(f(y)+g(y))| &\le |f(x)-f(y)|+|g(x)-g(y)|\\ &\le (c_1+c_2)|x-y| \quad (x, y \in [0,1])\end{aligned}$$

and so the function defined by $f(x)+g(x)$ is also Lipschitz.

For $x, y \in [0,1]$,

$$\begin{aligned}|f(x)g(x)-f(y)g(y)| &= |f(x)g(x)-f(y)g(x)+f(y)g(x)-f(y)g(y)|\\ &\le |f(x)g(x)-f(y)g(x)|+|f(y)g(x)-f(y)g(y)|\\ &= |g(x)||f(x)-f(y)|+|f(y)||g(x)-g(y)|.\end{aligned}$$

Moreover, for $x \in [0,1]$, we have $|f(x)-f(0)| \le c_1|x| \le c_1$, so that $|f(x)| \le |f(0)|+c_1 = c_1'$, say. Similarly $|g(x)| \le c_2'$. Thus

$$\begin{aligned}|f(x)g(x)-f(y)g(y)| &\le |c_1'||f(x)-f(y)|+|c_2'||g(x)-g(y)|\\ &\le (c_1c_1'+c_2c_2')|x-y|\end{aligned}$$

so $f(x)g(x)$ is Lipschitz.

1.13 Given $x, y \in \mathbb{R}$ with $y \ne x$, it follows from the mean-value theorem that there exists $a \in (y, x)$ or $a \in (x, y)$ with

$$\frac{f(x)-f(y)}{x-y} = f'(a).$$

Thus

$$\left|\frac{f(x)-f(y)}{x-y}\right| = |f'(a)| \le c$$

and hence

$$|f(x)-f(y)| \le c|x-y| \ (x, y \in \mathbb{R})$$

so that f is a Lipschitz function.

1.14 If $f : X \to Y$ is a Lipschitz function, then

$$|f(x)-f(y)| \le c|x-y| \quad (x, y \in \mathbb{R}),$$

for some $c > 0$. Thus, given $\epsilon > 0$ and $y \in \mathbb{R}$, it follows that $|f(x)-f(y)| < \epsilon$, whenever

$$c|x-y| < \epsilon,$$

that is, whenever

$$|x - y| < \epsilon/c.$$

So, on taking $\delta = \epsilon/c > 0$, it follows that f is continuous at y, using the 'epsilon-delta' definition of continuity.

1.15 Note that if $y = f(x) = x^2 + x$, then solving the quadratic equation for x, we get $x = \frac{1}{2}(-1 \pm (1 + 4y)^{1/2})$, taking real values only. Thus (i) $f^{-1}(2) = \{-2, 1\}$. (ii) $f^{-1}(-2) = \emptyset$. (iii) As y increases from 2 to 6, $(1 + 4y)^{1/2}$ increases from 3 to 5, so x runs over two ranges $[1, 2]$ and $[-3, -2]$. Hence $f^{-1}([2, 6]) = [-3, -2] \cup [1, 2]$.

1.16 For $0 \le x, y \le 2$,

$$|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \le 4|x - y|$$

so f is also Lipschitz on $[0, 2]$.

Thus f is also Lipschitz on $[1, 2]$, with $f([1, 2]) = [1, 4]$. For $1 \le x, y \le 4$,

$$|f^{-1}(x) - f^{-1}(y)| = |\sqrt{x} - \sqrt{y}| = \left|\frac{x - y}{\sqrt{x} + \sqrt{y}}\right| \le \frac{1}{2}|x - y|$$

so f^{-1} is Lipschitz on $[1, 4]$, so f is bi-Lipschitz on $[1, 2]$.

For $x > 0$,

$$\frac{|f(2x) - f(x)|}{|2x - x|} = \frac{4x^2 - x^2}{x} = 3x.$$

Thus $|f(x) - f(y)|/|x - y|$ is not bounded on $\mathbb{R}$ so f is not Lipschitz on $\mathbb{R}$.

1.17 We use the 'open cover' definition of compactness. Let E be compact, f continuous, and $f(E) \subset \bigcup U_i$, a cover of $f(E)$ by open sets. Since f is continuous, the sets $f^{-1}(U_i)$ are open, so $E \subset \bigcup f^{-1}(U_i)$ is a cover of E by open sets. By compactness of E this has a finite subcover, say $E \subset \bigcup_{r=1}^{m} f^{-1}(U_{i(r)})$, so $f(E) \subset \bigcup_{r=1}^{m} U_{i(r)}$, which gives a cover of $f(E)$ by a finite subset of the U_i. Thus $f(E)$ is compact.

1.18 We take complements in A_1. Thus $A_1 \setminus A_2, A_1 \setminus A_3, \ldots$ is an increasing sequence of sets, so by (1.6)

$$\mu\left(A_1 \setminus \bigcap_{i=1}^{\infty} A_i\right) = \mu\left(\bigcup_{i=1}^{\infty}(A_1 \setminus A_i)\right) = \lim_{i\to\infty} \mu(A_1 \setminus A_i).$$

Since $\mu(A_1) < \infty$, this gives $\mu(A_1) - \mu\left(\bigcap_{i=1}^{\infty} A_1\right) = \lim_{i\to\infty}(\mu(A_1) - \mu(A_i)) = \mu(A_1) - \lim_{i\to\infty} \mu(A_i)$, so $\mu\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{i\to\infty} \mu(A_i)$.

1.19 We show that μ satisfies conditions (1.1)–(1.4) and is hence a measure.

First, since $a \notin \emptyset$, $\mu(\emptyset) = 0$ and thus (1.1) is satisfied.

Secondly, suppose that $A \subset B$. If $a \in A$, then a also belongs to B and hence $\mu(A) = \mu(B) = 1$. If $a \notin A$, then $\mu(A) = 0 \leq \mu(B)$. Thus, in both of the two possible cases, (1.2) is satisfied.

Finally, suppose that $A_1, A_2, \ldots$ is a sequence of sets. If $a \notin A_i$, for each positive integer, then $a \notin \bigcup_{i=1}^{\infty} A_i$ so that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 0 = \sum_{i=1}^{\infty} \mu(A_i).$$

On the other hand, if $a \in A_j$, for some integer j, then $a \in \bigcup_{i=1}^{\infty} A_i$ so that

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 = \mu(A_j) \leq \sum_{i=1}^{\infty} \mu(A_i).$$

If the sets A_i are disjoint, then $a \notin A_i$ for $i \neq j$ so that $\mu(A_i) = 0$ for $i \neq j$ and hence

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 = \sum_{i=1}^{\infty} \mu(A_i).$$

Thus, in both of the two possible cases, (1.3) and (1.4) are satisfied.

1.20 With the construction of the middle third Cantor set F as indicated in figure 0.1, the kth stage of the construction E_k is the union of 2^k intervals each of length 3^{-k}, with $E_0 \supset E_1 \supset E_2 \supset \ldots$ and $F = \bigcap_{k=1}^{\infty} E_k$.

Define a mass distribution μ by starting with unit mass on $E_0 = [0, 1]$, splitting this equally between the two intervals of E_1, splitting the mass on each of these intervals equally between the two sub-intervals in E_2, etc. Thus we construct a mass distribution μ on F by repeated subdivision, splitting the mass in as uniform a way as possible at each stage. For each interval I in E_k we have $\mu(I) = 2^{-k}$, and this allows us to calculate the mass of any combination of intervals from the E_k and defines μ on every subset of $\mathbb{R}$.

1.21 For all $\epsilon > 0$, $\emptyset \subset [0, \epsilon]$ so $\mathcal{L}^1(\emptyset) \leq \mathcal{L}^1([0, \epsilon]) = \epsilon$. This is true for arbitrarily small $\epsilon > 0$, so $\mathcal{L}(\emptyset) = 0$, as required for (1.1).

Let $A \subset B$. Given $\epsilon > 0$ we may find a countable collection of intervals $[a_i, b_i]$ such that $A \subset B \subset \bigcup_{i=1}^{\infty}[a_i, b_i]$ with $\sum_{i=1}^{\infty}(b_i - a_i) < \mathcal{L}^1(B) + \epsilon$.

It follows that $\mathcal{L}^1(A) \le \mathcal{L}^1(B) + \epsilon$ for all $\epsilon > 0$, so that $\mathcal{L}^1(A) \le \mathcal{L}^1(B)$ for (1.2).

For (1.3), assume that $\mathcal{L}^1(A_i) < \infty$ for each i, since the result is clearly true otherwise. For each $\epsilon > 0$ and $i = 1, 2, \ldots$, there exist intervals $[a_{i,j}, b_{i,j}]$ such that

$$A_i \subset \bigcup_{j=1}^{\infty}[a_{i,j}, b_{i,j}] \text{ and } \sum_{j=1}^{\infty}(b_{i,j} - a_{i,j}) < \mathcal{L}^1(A_i) + \frac{\epsilon}{2^i}.$$

Clearly $\bigcup_{i=1}^{\infty} A_i \subset \bigcup_{i=1}^{\infty} \bigcup_{j=1}^{\infty}[a_{i,j}, b_{i,j}]$ and so

$$\mathcal{L}^1\left(\bigcup_{i=1}^{\infty} A_i\right) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}(b_{i,j} - a_{i,j}) \le \sum_{i=1}^{\infty}\left(\mathcal{L}^1(A_i) + \frac{\epsilon}{2^i}\right) = \sum_{i=1}^{\infty}\mathcal{L}^1(A_i) + \epsilon.$$

It follows that (1.3) holds.

1.22 We begin by showing that μ satisfies conditions (1.1)–(1.4) and is hence a measure.

First,

$$\mu(\emptyset) = \mathcal{L}^1(\{x : (x, f(x)) \in \emptyset\}) = \mathcal{L}^1(\emptyset) = 0$$

and so (1.1) is satisfied.

Second, if $A \subset B$, then $\{x : (x, f(x)) \in A\} \subset \{x : (x, f(x)) \in B\}$ and so, since $\mathcal{L}^1$ is a measure,

$$\mu(A) = \mathcal{L}^1(\{x : (x, f(x)) \in A\}) \le \mathcal{L}^1(\{x : (x, f(x)) \in B\}) = \mu(B)$$

so that (1.2) is satisfied.

Finally, if $A_1, A_2, \ldots$ is a sequence of sets, then, since $\mathcal{L}^1$ is a measure,

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mathcal{L}^1\left(\left\{x : (x, f(x)) \in \bigcup_{i=1}^{\infty} A_i\right\}\right)\\ &= \mathcal{L}^1\left(\bigcup_{i=1}^{\infty}\{x : (x, f(x)) \in A_i\}\right)\\ &\le \sum_{i=1}^{\infty}\mathcal{L}^1(\{x : (x, f(x)) \in A_i\}) = \sum_{i=1}^{\infty}\mu(A_i)\end{aligned}$$

so that (1.3) is satisfied. If the sets A_i are disjoint Borel sets then, since $\mathcal{L}^1$ is a measure,

$$\begin{aligned}\mu\left(\bigcup_{i=1}^{\infty} A_i\right) &= \mathcal{L}^1\left(\left\{x : (x, f(x)) \in \bigcup_{i=1}^{\infty} A_i\right\}\right)\\ &= \mathcal{L}^1\left(\bigcup_{i=1}^{\infty}\{x : (x, f(x)) \in A_i\}\right)\\ &= \sum_{i=1}^{\infty} \mathcal{L}^1(\{x : (x, f(x)) \in A_i\}) = \sum_{i=1}^{\infty} \mu(A_i)\end{aligned}$$

so that (1.4) is satisfied.

Thus μ is a measure on $\mathbb{R}^2$. We now show that μ is supported by the graph of f. We begin by noting that, since $[0, 1]$ is compact (that is, closed and bounded) and the map F defined by $F(x) = (x, f(x))$ is continuous, then the graph of f which is equal to $F([0, 1])$ is also compact and hence closed. Clearly,

$$\mu(\mathbb{R}^2\backslash \mathrm{graph} f) = \mathcal{L}^1(\{x : (x, f(x)) \in \mathbb{R}^2\backslash \mathrm{graph} f\}) = \mathcal{L}^1(\emptyset) = 0.$$

Now let $a \in [0, 1]$ and let $r > 0$. Since f is continuous, a belongs to a non-trivial interval $I_r \subset [0, 1]$ such that, for each $x \in I_r$, we have $(x, f(x)) \in B((a, f(a)), r)$ and hence

$$\mu(B((a, f(a)), r)) = \mathcal{L}^1(\{x : (x, f(x)) \in B((a, f(a)), r)\}) \geq \mathcal{L}^1(I_r) > 0.$$

Thus graph f is the smallest closed set X such that $\mu(\mathbb{R}^2\backslash X) > 0$; that is, graph f is the support of μ.

Finally,

$$\mu(\mathrm{graph} f) = \mathcal{L}^1([0, 1]) = 1$$

so that $0 < \mu(\mathrm{graph} f) < \infty$ and hence μ is a mass distribution.

1.23 For positive integers m, n define sets

$$A_{m,n} = \left\{x \in D : |f_k(x) - f(x)| < \frac{1}{m} \text{ for all } k \geq n\right\}.$$

For each m the sequence of sets $A_{m,1} \subset A_{m,2} \subset A_{m,2} \subset \ldots$ is increasing with $\bigcup_{n=1}^{\infty} A_{m,n} = D$, so by (1.6) there is a positive integer n_m such that

$\mu(D\backslash A_{m,n}) < 2^{-m}\epsilon$ for all $n \geq n_m$. Define $A = \bigcap_{m=1}^{\infty} A_{m,n_m}$. Then

$$\mu(D \setminus A) \leq \mu\left(\bigcup_{m=1}^{\infty} D\backslash A_{m,n_m}\right) \leq \sum_{m=1}^{\infty} \frac{\epsilon}{2^m} \leq \epsilon.$$

Let $\delta > 0$ and take $m > 1/\delta$. If $x \in A$, then $x \in A_{m,n_m}$, so $|f_k(x) - f(x)| < \frac{1}{m} < \delta$ for all $k \geq n_m$, so $f_k(x) \to f(x)$ uniformly on A.

1.24 For $n = 1, 2, \ldots$ let $D_n = \{x : f(x) \geq 1/n\}$. Then

$$0 = \int_D f d\mu \geq \int_D \frac{1}{n}\chi_{D_n} d\mu = \frac{1}{n}\mu(D_n),$$

since $\frac{1}{n}\chi_{D_n}$ is a simple function. Thus $\mu(D_n) = 0$ for all n. Since $\{x : f(x) > 0\} = \bigcup_{n=1}^{\infty} D_n$, it follows that $\mu\{x : f(x) > 0\} = 0$, that is $f(x) = 0$ for almost all x.

1.25 $\mathsf{E}((X - \mathsf{E}(X))^2) = \mathsf{E}(X^2 - 2X\mathsf{E}(X) + \mathsf{E}(X)^2) = \mathsf{E}(X^2) - 2\mathsf{E}(X)\mathsf{E}(X) + \mathsf{E}(X)^2 = \mathsf{E}(X^2) - \mathsf{E}(X)^2$.

1.26 The uniform distribution on $[a, b]$ has p.d.f. $f(u) = 1/(a - b)$ for $a \leq u \leq b$ and $f(u) = 0$ otherwise. Thus

$$\begin{aligned}\mathsf{E}(X) &= (a-b)^{-1}\int_a^b u du = (a-b)^{-1}\left[\frac{1}{2}u^2\right]_a^b \\ &= \frac{1}{2}(b^2 - a^2)/(b-a) = \frac{1}{2}(a+b). \\ \mathsf{E}(X^2) &= (a-b)^{-1}\int_a^b u^2 du = (a-b)^{-1}\left[\frac{1}{3}u^3\right]_a^b \\ &= \frac{1}{3}(b^3 - a^3)/(b-a) = \frac{1}{3}(a^2 + ab + b^2).\end{aligned}$$

Thus

$$\begin{aligned}\mathrm{var}(X) &= E(X^2) - E(X)^2 = \frac{1}{3}(a^2 + ab + b^2) - \frac{1}{4}(a+b)^2 \\ &= \frac{1}{12}(a^2 - 2ab + b^2) = \frac{1}{12}(a-b)^2.\end{aligned}$$

1.27 Define random variables X_k by $X_k = 0$ if $\omega \notin A_k$ and $X_k = 1$ if $\omega \in A_k$. Then $N_k = X_1 + \cdots + X_k$, so by the strong law of large numbers (1.25), $N_k/k \to \mathsf{E}(X_k) = p$. Thus taking A_k to be the event that the kth trial is successful, N_k/k is the proportion of successes, which converges to p, the probability of success.

1.28 With $X_k = 1$ if a six is scored on he kth throw and 0 otherwise, and $S_k = X_1 + \cdots + X_k$ as the number of sixes in the first k throws, X_k has mean $m = \frac{1}{6}$ and variance $\sigma^2 = \frac{1}{6}\left(\frac{5}{6}\right)^2 + \frac{5}{6}\left(\frac{1}{6}\right)^2 = \frac{5}{36}$. By (1.26)

$$\mathsf{P}(S_k \geq 1050) = \mathsf{P}\left(\frac{S_k - 1000}{\sqrt{5/36}\sqrt{6000}} \geq \sqrt{3}\right)$$
$$\simeq \int_{\sqrt{3}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}u^2\right) du = 0.075.$$

Chapter 2

2.1 Put

$$\overline{\mathcal{H}}^s_\delta(F) = \inf\left\{\sum_i |U_i|^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \text{ by closed sets}\right\}.$$

Since we have reduced the class of permissible covers by restricting to covers by closed sets, we must have $\overline{\mathcal{H}}^s_\delta(F) \geq \mathcal{H}^s_\delta(F)$. Now suppose that $\{U_i\}$ is a δ-cover of F. Since the closure $\overline{U}_i$ of U_i satisfies $|\overline{U}_i| = |U_i|$, it follows that $\{\overline{U}_i\}$ is a δ-cover of F by closed sets with $\sum_i |\overline{U}_i|^s = \sum_i |U_i|^s$. Since this is true for every δ-cover of F, it follows that $\overline{\mathcal{H}}^s_\delta(F) \leq \mathcal{H}^s_\delta(F)$. Thus $\overline{\mathcal{H}}^s_\delta(F) = \mathcal{H}^s_\delta(F)$ for all $\delta > 0$ and so the value of $\mathcal{H}^s(F) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(F)$ is unaltered if we only consider δ-covers by closed sets.

2.2 Suppose that $\{U_i\}$ is a δ-cover of F. For any set U_i in the cover we have $|U_i|^0 = 1$ and so $\sum_i |U_i|^0$ is equal to the number of sets in the cover. Thus $\mathcal{H}^0_\delta(F)$ is the smallest number of sets that form a δ-cover of F.

If F has k points, $x_1, x_2, \ldots, x_k$, then the k balls of radius $\delta/2$ with centers at $x_1, x_2, \ldots, x_k$ form a δ-cover of F and so $\mathcal{H}^0_\delta(F) \leq k$. Moreover, if $\delta > 0$ is so small that $|x_i - x_j| > \delta$ for all $i \neq j$, then any δ-cover of F must contain at least k sets and so $\mathcal{H}^0_\delta(F) \geq k$. So, for δ small enough, we have $\mathcal{H}^0_\delta(F) = k$ and hence $\mathcal{H}^0(F) = \lim_{\delta\to 0}\mathcal{H}^0_\delta(F) = k$.

Finally, if F has infinitely many points, then for each positive integer k, we can take a set $F_k \subset F$ such that F_k has k points. Then $\mathcal{H}^0(F) \geq \mathcal{H}^0(F_k) = k$ for all k, so $\mathcal{H}^0(F) = \infty$.

2.3 Clearly, for every $0 < \epsilon \leq \delta$, we may cover the empty set with a single set of diameter ϵ, so $0 \leq \mathcal{H}^s_\delta(\emptyset) \leq \epsilon^s$ for all $\epsilon > 0$, giving $\mathcal{H}^s_\delta(\emptyset) = 0$. Thus $\mathcal{H}^s(F) = \lim_{\delta\to 0}\mathcal{H}^s_\delta(F) = 0$.

If $E \subset F$, every δ-cover of F is also a δ-cover of E, so taking the infimum over all δ-covers gives $\mathcal{H}^s_\delta(E) \leq \mathcal{H}^s_\delta(F)$ for all $\delta > 0$. Letting $\delta \to 0$ gives $\mathcal{H}_\delta(E) \leq \mathcal{H}_\delta(F)$.

Now let $F_1, F_2, \ldots$ be subsets of $\mathbb{R}^n$. Without loss of generality, we may assume that $\sum_{i=1}^{\infty} \mathcal{H}_\delta^s(F_i) < \infty$. For $\epsilon > 0$ let $\{U_{i,j} : j = 1, 2, \ldots\}$ be a δ-cover of F_i such that $\sum_{j=1}^{\infty} |U_{i,j}|^s \le \mathcal{H}_\delta^s(F_i) + 2^{-i}\epsilon$. Then $\{U_{i,j} : i = 1, 2, \ldots, j = 1, 2, \ldots\}$ is a δ-cover of $\bigcup_{i=1}^{\infty} F_i$ and

$$\mathcal{H}_\delta^s\left(\bigcup_{i=1}^{\infty} F_i\right) \le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} |U_{i,j}|^s \le \sum_{i=1}^{\infty}\left(\mathcal{H}_\delta^s(F_i) + \frac{\epsilon}{2^i}\right) = \epsilon + \sum_{i=1}^{\infty} \mathcal{H}_\delta^s(F_i)$$
$$\le \epsilon + \sum_{i=1}^{\infty} \mathcal{H}^s(F_i).$$

Since this is true for every $\epsilon > 0$, it follows that

$$\mathcal{H}^s\left(\bigcup_{i=1}^{\infty} F_i\right) = \lim_{\delta\to 0} \mathcal{H}_\delta^s\left(\bigcup_{i=1}^{\infty} F_i\right) \le \sum_{i=1}^{\infty} \mathcal{H}^s(F_i)$$

as required.

2.4 Note that in calculating $\mathcal{H}_\delta^s([0, 1])$ it is enough to consider coverings by intervals.

If $0 \le s < 1$ and $\{U_i\}$ is a δ-cover of $[0, 1]$ by intervals, then

$$1 \le \sum_i |U_i| = \sum_i |U_i|^{1-s}|U_i|^s \le \delta^{1-s}\sum_i |U_i|^s.$$

Hence $\mathcal{H}_\delta^s([0, 1]) \ge \delta^{s-1}$, so letting $\delta \to 0$ gives $\mathcal{H}^s([0, 1]) = \infty$.

For $s > 1$, we may cover $[0, 1]$ by at most $(1 + 1/\delta)$ intervals of length δ, so

$$\mathcal{H}_\delta^s([0, 1]) \le (1 + 1/\delta)\delta^s \to 0$$

as $\delta \to 0$, so $\mathcal{H}^s([0, 1]) = 0$.

For $s = 1$, if $\{U_i\}$ is a δ-cover of $[0, 1]$ by intervals, then $1 \le \sum_i |U_i|$, so $\mathcal{H}_\delta^1([0, 1]) \ge 1$, and letting $\delta \to 0$ gives $\mathcal{H}^1([0, 1]) \ge 1$.

Taking a cover $[0, 1]$ by at most $(1 + 1/\delta)$ intervals of length δ,

$$\mathcal{H}_\delta^1([0, 1]) \le (1 + 1/\delta)\delta \to 1$$

as $\delta \to 0$, so $\mathcal{H}^1([0, 1]) \ge 1$. We conclude that $\mathcal{H}^1([0, 1]) = 1$.

2.5 First suppose that F is bounded, say $F \subset [-m, m]$. By the mean value theorem, for some $z \in [-m, m]$,

$$|f(x) - f(y)| = |f'(z)||x - y| \le \left(\sup_{z \in [-m,m]} |f'(z)| \right) |x - y|$$

Since $f'(z)$ is continuous it is bounded on $[-m, m]$. Thus f is Lipschitz on F, so $\dim_H f(F) \le \dim_H(F)$ by Corollary 2.4. For arbitrary $F \subset \mathbb{R}$, $f(F) = \bigcup_{m=1}^{\infty} f(F_n \cap [-m, m])$, so by countable stability

$$\begin{aligned} \dim_H f(F) &= \sup_m \dim_H f(F_n \cap [-m, m]) \le \sup_m \dim_H (F_n \cap [-m, m]) \\ &\le \dim_H F, \end{aligned}$$

by the bounded case.

2.6 Let $F_k = F \cap [1/k, k]$. If $x, y \in F_k$, then

$$|f(x) - f(y)| = |x^2 - y^2| = |x - y||x + y|$$

and so

$$\frac{2}{k}|x - y| \le |f(x) - f(y)| \le 2k|x - y|.$$

Thus f is a bi-Lipschitz map on F_k and so, by Corollary 2.4, $\dim_H f(F_k) = \dim_H F_k$. Similarly, if $G_k = F \cap [-k, -1/k]$, then $\dim_H f(G_k) = \dim_H G_k$.

Now $F = (F \cap \{0\}) \cup \bigcup_{k=1}^{\infty} (F_k \cup G_k)$ and $f(F) = f(F \cap \{0\}) \cup \bigcup_{k=1}^{\infty} (f(F_k) \cup f(G_k))$. Since $F \cap \{0\}$ and $f(F \cap \{0\})$ contain at most one point, they both have zero dimension. Thus, by countable stability,

$$\begin{aligned} \dim_H F &= \sup\{\dim_H F_k, \dim_H G_k : k = 1, 2, \ldots\} \\ &= \sup\{\dim_H f(F_k), \dim_H f(G_k) : k = 1, 2, \ldots\} = \dim_H f(F). \end{aligned}$$

[Note that this result is not true for box dimension. For example, using Example 3.5 and Exercise 3.11 we see that $\dim_B f(F) \ne \dim_B F$ when $F = \{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$.]

2.7 Define $g : [0, 1] \to \text{graph} f$ by $g(x) = (x, f(x))$. We claim that g is bi-Lipschitz. For:

$$|g(x) - g(y)|^2 = |x - y|^2 + |f(x) - f(y)|^2$$

so

$$|x - y|^2 \le |g(x) - g(y)|^2 \le |x - y|^2 + c^2|x - y|^2 = (1 + c^2)|x - y|^2$$

since $|f(x) - f(y)| \le c|x - y|$ for some $c > 0$. Thus g is bi-Lipschitz, so $1 = \dim_H([0, 1]) = \dim_H g([0, 1]) = \dim_H \text{graph} f$.

2.8 Both $\{0, 1, 2, 3, \ldots\}$ and $\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ are countable sets, so have Hausdorff dimension 0.

2.9 Note that F splits into 9 parts $F_i = F \cap [i/10, (i+1)/10]$ for $i = 0, 1, 2, 3, 4, 6, 7, 8, 9$, these parts disjoint except possibly for endpoints which have s-dimensional measure 0 if $s > 0$. It follows from Scaling property 2.1 that, for $s > 0$, $\mathcal{H}^s(F_i) = 10^{-s}\mathcal{H}^s(F)$ for all i. Summing, and using that F is essentially a disjoint union of the F_i, it follows that, for $s > 0$,

$$\mathcal{H}^s(F) = \sum_{i=0,1,2,3,4,6,7,8,9} \mathcal{H}^s(F_i) = 9 \times 10^{-s}\mathcal{H}^s(F).$$

If we assume that when $s = \dim_{\mathrm{H}} F$ we have $0 < \mathcal{H}^s(F) < \infty$, then, for this value of s, we may divide through by $\mathcal{H}^s(F)$ to obtain $1 = 9 \times 10^{-s}$ and hence $s = \dim_{\mathrm{H}} F = \log 9/\log 10 = 0.954\ldots$.

2.10 Note that, for $i, j = 0, 1, 2, 3, 4, 6, 7, 8, 9$ the sets $F \cap ([i/10, (i+1)/10] \times [j/10, (j+1)/10])$ are scale $1/10$ similar copies of F. By the addition and scaling properties of Hausdorff measure,

$$\begin{aligned}\mathcal{H}^s(F) &= \sum_{i,j\neq 5} \mathcal{H}^s(F \cap ([i/10, (i+1)/10] \times [j/10, (j+1)/10]) \\ &= 9^2 10^{-s}\mathcal{H}^s(F),\end{aligned}$$

provided $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_{\mathrm{H}} F$, in which case $1 = 9^2 10^{-s}$, giving $s = 2\log 9/\log 10 = 1.908\ldots$.

2.11 The set F comprises one similar copy of itself at scale $\frac{1}{2}$, say F_0, and four similar copies at scale $\frac{1}{4}$, say F_1, F_2, F_3, F_4. By the additive and scaling properties of Hausdorff measure, noting that the F_i intersect only in single points,

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_0) + \sum_{i=1}^{4} \mathcal{H}^s(F_i) = \left(\frac{1}{2}\right)^s \mathcal{H}^s(F) + 4\left(\frac{1}{4}\right)^s \mathcal{H}^s(F)$$

for $s > 0$. Provided $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_{\mathrm{H}} F$, we have $1 = \left(\frac{1}{2}\right)^s + 4\left(\frac{1}{2}\right)^s$. Thus $4\left(\frac{1}{2}^s\right)^2 + \left(\frac{1}{2}\right)^s - 1 = 0$; solving this quadratic equation in $\left(\frac{1}{2}\right)^s$ gives $\left(\frac{1}{2}\right)^s = (-1 + \sqrt{17})/8$ as the positive solution, so $s = (\log 8 - \log(\sqrt{17} - 1))/\log 2 = 1.357\ldots$.

2.12 F is the union of countably many translates of the middle third Cantor set, all of which have Hausdorff dimension $\log 2/\log 3$, so $\dim_{\mathrm{H}} F = \log 2/\log 3 = 0.6309\ldots$ using countable stability.

2.13 F is the union, over all finite sequences $a_1, a_2, \dots, a_k$ of the digits 0, 1, 2, of similar copies of the middle third Cantor set scaled by a factor 3^{-k} and translated to have left hand end at $0.a_1a_2\dots a_k$, to base 3. Thus F is the union of countably many similar copies of the Cantor set, so $\dim_H F = \log 2/\log 3$ using countable stability.

2.14 The set F is the union of two disjoint similar copies of itself, F_L, F_R, say, at scales $\frac{1}{2}(1-\lambda)$. By the additive and scaling properties of Hausdorff measure

$$\mathcal{H}^s(F) = \mathcal{H}^s(F_L) + \mathcal{H}^s(F_R) = 2\left(\frac{1}{2}(1-\lambda)\right)^s \mathcal{H}^s(F)$$

for $s \geq 0$. Provided $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_H F$, we have $1 = 2\left(\frac{1}{2}(1-\lambda)\right)^s$, giving that $\dim_H F = \log 2/\log(2/(1-\lambda))$.

The set E is the union of four disjoint similar copies of itself, E_1, E_2, E_3, E_4, say, at scales $\frac{1}{2}(1-\lambda)$. By the additive and scaling properties of Hausdorff measure

$$\mathcal{H}^s(F) = \sum_{i=1}^{4} \mathcal{H}^s(F_i) = 4\left(\frac{1}{2}(1-\lambda)\right)^s \mathcal{H}^s(F)$$

for $s \geq 0$. Provided $0 < \mathcal{H}^s(F) < \infty$ when $s = \dim_H F$, we have $1 = 4\left(\frac{1}{2}(1-\lambda)\right)^s$, giving that $\dim_H F = \log 4/\log(2/(1-\lambda)) = 2\log 2/\log(2/(1-\lambda))$.

2.15 Take the unit square E_0 and divide it into 16 squares of side 1/4. Now take $0 < r < 1/4$, put a square of side r in the middle of each of the 16 small squares and discard everything that is not inside one of these squares, to get a set E_1.

Keep on repeating this process so that, at the k-th stage, there is a collection E_k of 16^k disjoint squares of side r^k. Then $F_r = \bigcap_k E_k$ is a totally disconnected subset of $\mathbb{R}^2$. (If two points x, y are in the same component of F_r, then they must belong to the same square in E_k, for all $k = 1, 2, \dots$. Thus $|x-y| \leq \sqrt{2}r^k$, for each $k = 1, 2, \dots$, and hence $|x-y| = 0$ so that $x = y$.)

The set F_r is made up of 16 disjoint similar copies of itself, each scaled by a factor r, denote these sets as $F_{r,1}, \dots, F_{r,16}$. It follows from Scaling property 2.1 that, for $s \geq 0$,

$$\mathcal{H}^s(F_r) = \sum_{i=1}^{16} \mathcal{H}^s(F_{r,i}) = \sum_{i=1}^{16} r^s \mathcal{H}^s(F_r).$$

Assuming that when $s = \dim_{\mathrm{H}} F_r$ we have $0 < \mathcal{H}^s(F_r) < \infty$ (using the heuristic method), then, for this value of s we may divide by $\mathcal{H}^s(F_r)$ to obtain $1 = 16r^s$ and so $s = \dim_{\mathrm{H}} F_r = -\log 16/\log r$. As r increases from 0 to 1/4, $\dim_{\mathrm{H}} F_r$ increases from 0 to 2, taking every value in between.

A set consisting of a single point gives a totally disconnected subset of $\mathbb{R}^2$ of Hausdorff dimension zero. It remains to show that there exists a totally disconnected subset of $\mathbb{R}^2$ of Hausdorff dimension two. For one of many ways to do this, let $G = \bigcup_{k=5}^{\infty} G_k$, where $G_k = F_{1/4-1/k} + (k, 0)$. The sets G_k are disjoint and hence G is a totally disconnected subset of $\mathbb{R}^2$. By countable stability, we have

$$\dim_{\mathrm{H}} G = \sup_{5 \le k \le \infty} \dim_{\mathrm{H}} G_k = \sup_{5 \le k \le \infty} \dim_{\mathrm{H}} (F_{1/4-1/k}) = 2,$$

using that G_k is congruent to $F_{1/4-1/k}$, whose dimension tends to 2 as $k \to \infty$.

2.16 Note that F is just a copy of the middle third Cantor set scaled by $\frac{1}{3}\pi$. Thus $\dim_{\mathrm{H}} F = \log 2/\log 3 = 0.6309\ldots$.

2.17 We use the notation of Section 2.5. Let U_i be a δ-cover of F. Then

$$\sum h(|U_i|) = \sum (h(|U_i|)/g(|U_i|))g(|U_i|) \le \eta(\delta) \sum g(|U_i|)$$

where $\eta(\delta) = \sup_{0<t\le\delta} h(t)/g(t)$. Taking infima, $\mathcal{H}^h_\delta(F) \le \eta(\delta)\mathcal{H}^g_\delta(F)$. Letting $\delta \to 0$, then $\eta(\delta) \to 0$, and $\mathcal{H}^g_\delta(F) \to \mathcal{H}^g(F) < \infty$, so $\mathcal{H}^h_\delta(F) \to 0$, that is $\mathcal{H}^h(F) = 0$.

Chapter 3

3.1 Suppose that F can be covered by $N_\delta(F)$ sets of diameter at most δ. Then, by the Lipschitz condition, the $N_\delta(F)$ images of these sets under f form a cover of $f(F)$ by sets of diameter at most $c\delta$. So, considering values of δ for which $c\delta < 1$, we have

$$\begin{aligned}\overline{\dim}_{\mathrm{B}} f(F) &= \overline{\lim}_{c\delta\to 0} \frac{\log N_{c\delta}(f(F))}{-\log c\delta} \le \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta - \log c} \\ &= \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} = \overline{\dim}_{\mathrm{B}} F.\end{aligned}$$

Note that we could replace upper limits by lower limits throughout the above argument to get the corresponding result for lower box dimensions.

Now suppose that f satisfies the Hölder condition

$$|f(x) - f(y)| \leq c|x - y|^\alpha \quad (x, y \in F).$$

Suppose that F can be covered by $N_\delta(F)$ sets of diameter at most δ. Then the $N_\delta(F)$ images of these sets under f form a cover of $f(F)$ by sets of diameter at most $c\delta^\alpha$. Thus

$$\begin{aligned}\overline{\dim}_{\mathrm{B}} f(F) &= \overline{\lim}_{c\delta^\alpha \to 0} \frac{\log N_{c\delta^\alpha}(f(F))}{-\log c\delta^\alpha} \leq \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\alpha \log \delta - \log c} \\ &= \frac{1}{\alpha} \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} = \frac{1}{\alpha} \overline{\dim}_{\mathrm{B}} F.\end{aligned}$$

Again, a similar argument gives the result for lower box dimensions.

3.2 Let F be a subset of $\mathbb{R}^n$, let $N_\delta(F)$ denote the smallest number of closed balls of radius δ that cover F and let $N'_\delta(F)$ denote the number of δ-mesh cubes that intersect F.

For each δ-mesh cube that intersects F, take a closed ball of radius $\delta\sqrt{n}$ whose centre is at the centre of the cube; the ball clearly contains the cube (whose diagonal is of length $\delta\sqrt{n}$) and so $N_{\delta\sqrt{n}}(F) \leq N'_\delta(F)$. On the other hand, any closed ball of radius δ intersects at most 4^n δ-mesh cubes and so $N'_\delta(F) \leq 4^n N_\delta(F)$. Combining:

$$N_{\delta\sqrt{n}}(F) \leq N'_\delta(F) \leq 4^n N_\delta(F)$$

so that if $\delta\sqrt{n} < 1$, then

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log \delta} \leq \frac{\log N'_\delta(F)}{-\log \delta} \leq \frac{\log 4^n N_\delta(F)}{-\log \delta}$$

so

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log \delta\sqrt{n} + \log \sqrt{n}} \leq \frac{\log N'_\delta(F)}{-\log \delta} \leq \frac{\log 4^n + \log N_\delta(F)}{-\log \delta}.$$

Taking lower limits as $\delta \to 0$, so that also $\delta\sqrt{n} \to 0$, we get that

$$\underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta} \leq \underline{\lim}_{\delta \to 0} \frac{\log N'_\delta(F)}{-\log \delta} \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta},$$

so these terms are equal; in other words the value of the expression for lower box-counting dimension is the same for both $N_\delta(F)$ (using definition (i) of lower box dimension), and $N'_\delta(F)$ (using definition (iii)).

The correspondence of the two definitions of upper box dimension follows in exactly the same way but taking upper limits.

3.3 Let E_k denote those numbers in $[0, 1]$ whose expansions do not contain the digit 5 in the first k decimal places. Then $F = \bigcap_{k=1}^{\infty} E_k$. Let $N_\delta(F)$ denote the least number of intervals of length δ that can cover F. Let k be the integer such that $10^{-k} \leq \delta < 10^{-k-1}$. Since E_k may be regarded as the union of 9^k intervals of lengths 10^{-k}, we get $N_\delta(F) \leq 9^k$, so

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \leq \overline{\lim}_{k\to\infty} \frac{\log 9^k}{-\log 10^{-k-1}}$$
$$\leq \overline{\lim}_{k\to\infty} \frac{k\log 9}{(k+1)\log 10} = \frac{\log 9}{\log 10}.$$

Now let $0 < \delta < 1$ and let k be the integer such that $10^{-k+1} \leq \delta < 10^{-k}$. Since any set of diameter δ can intersect at most two of the component intervals of E_k of length 10^{-k} and each such component interval contains points of F, at least $\frac{1}{2}9^k$ intervals of length δ are needed to cover F. Thus $N_\delta(F) \geq \frac{1}{2}9^k$, so

$$\underline{\dim}_{\mathrm{B}} F = \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta} \geq \underline{\lim}_{k\to\infty} \frac{\log \frac{1}{2}9^k}{-\log 10^{-k+1}}$$
$$\geq \underline{\lim}_{k\to\infty} \frac{k\log 9 - \log 2}{(k-1)\log 10} = \frac{\log 9}{\log 10} = 0.954\ldots.$$

We conclude that the box dimension of F exists, with $\dim_{\mathrm{B}} F = \log 9/\log 10$.

3.4 Let $N_\delta(F)$ denote the smallest number of squares (that is, 2-dimensional cubes) of side δ that cover F. We will use the fact (see after Equivalent definitions 3.1) that, if $\delta_k = 4^{-k}$, then

$$\dim_{\mathrm{B}} F = \lim_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log\delta_k},$$

if this limit exists.

It follows from the construction of F shown in figure 0.4 that $N_{\delta_k}(F) \leq 4^k$ and so

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log\delta_k} \leq \overline{\lim}_{k\to\infty} \frac{\log 4^k}{\log 4^k} = 1.$$

On the other hand, any square of side $\delta_k = 4^{-k}$ intersects at most two of the squares of side δ_k in E_k. Since F meets every one of the 4^k squares which comprise E_k, it follows that $N_{\delta_k}(F) \geq \frac{1}{2}4^k$, so

$$\underline{\dim}_{\mathrm{B}} F = \underline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log\delta_k} \geq \underline{\lim}_{k\to\infty} \frac{\log \frac{1}{2}4^k}{\log 4^k}$$
$$= \underline{\lim}_{k\to\infty} \frac{k\log 4 - \log 2}{k\log 4} = 1.$$

Thus $\dim_{\mathrm{B}} F = 1$.

3.5 Let $N_\delta(F)$ denote the smallest number of sets of diameter at most δ that cover F and let $\delta_k = 3^{-k}$. For each of the straight line segments that makes up E_k, take a closed disc of diameter δ_k, centred at the midpoint of the line. There are 4^k such discs and they cover F, so that (see Equivalent definitions 3.1 and comments following)

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \le \overline{\lim}_{k\to\infty} \frac{\log 4^k}{\log 3^k} = \frac{\log 4}{\log 3}.$$

Now let $N_\delta(F)$ denote the largest number of disjoint balls of radius δ with centres in F. The 4^k straight line segments that make up E_k have $4^k + 1$ distinct endpoints, each of which belongs to F. Balls of radius $1/3^{k+1}$ centred at these endpoints are mutually disjoint and so, putting $\delta_k = 3^{-(k+1)}$, we have by Equivalent definition (v), that

$$\begin{aligned}\underline{\dim}_{\mathrm{B}} F &= \underline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \ge \underline{\lim}_{k\to\infty} \frac{\log(4^k+1)}{\log 3^{k+1}} \\ &\ge \underline{\lim}_{k\to\infty} \frac{\log(4^k)}{\log 3^{k+1}} = \underline{\lim}_{k\to\infty} \frac{k\log 4}{(k+1)\log 3} = \frac{\log 4}{\log 3} = 1.262.\end{aligned}$$

3.6 Let $N_\delta(F)$ denote the smallest number of squares (that is, 2-dimensional cubes) of side δ that cover F. For $k = 1, 2, \ldots$, the Sierpiński triangle F can be covered by 3^k squares of side 2^{-k} and so, putting $\delta_k = 2^{-k}$, we have

$$\overline{\dim}_{\mathrm{B}} F = \overline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \le \overline{\lim}_{k\to\infty} \frac{\log 3^k}{\log 2^k} = \frac{\log 3}{\log 2}.$$

Now let $N_\delta(F)$ denote the largest number of disjoint balls of radius δ with centres in F. The top vertex of each of the 3^k triangles in E_k belongs to F and balls of radius $1/2^{k+1}$ centered at these vertices are mutually disjoint. So, putting $\delta_k = 2^{-(k+1)}$, we have

$$\underline{\dim}_{\mathrm{B}} F = \underline{\lim}_{k\to\infty} \frac{\log N_{\delta_k}(F)}{-\log \delta_k} \ge \underline{\lim}_{k\to\infty} \frac{\log 3^k}{\log 2^{k+1}} = \frac{\log 3}{\log 2}.$$

Thus $\dim_{\mathrm{B}} F = \log 3/\log 2 = 1.585\ldots$.

3.7 The middle third Cantor set has 2^k gaps of length 3^{-k-1} for $k = 0, 1, 2, \ldots$. If $\frac{1}{2}3^{-k} < \delta \le \frac{1}{2}3^{-k-1}$ the δ-neighbourhood fills the gaps of lengths 3^{-k} or less, and has two parts of length δ in the gaps of length 3^{-k-1} or more. Summing these lengths over all gaps, and noting that the parts of F_δ at each end of F have length δ, gives

$$\begin{aligned}\mathcal{L}(F_\delta) &= \sum_{i=k}^{\infty} 2^{i-1}3^{-i} + 2\delta \sum_{i=1}^{k-1} 2^{i-1} + 2\delta \\ &= \left(\frac{2}{3}\right)^{k-1} + 2^k\delta\end{aligned}$$

on summing the geometric series. Hence

$$2^k\delta \leq \mathcal{L}(F_\delta) \leq 2^k\delta + \left(\frac{2}{3}\right)^{k-1} \leq 4 \times 2^k\delta$$

or

$$c_1\delta^{1-\log 2/\log 3}\delta \leq \mathcal{L}(F_\delta) \leq c_2\delta^{1-\log 2/\log 3}.$$

Hence Proposition 3.2 gives that $\dim_{\mathrm{B}} F = \log 2/\log 3$

3.8 The idea is to construct a set such at some scales a relatively large number of boxes are needed in a covering and at other scales one can manage with relatively few. We adapt the middle third Cantor set by deleting the middle 3/5 of intervals at certain scales rather than the middle 1/3. Thus set $k_n = 10^n$, for $n = 0, 1, 2, \ldots$ and let $E = \bigcap_{k=0}^{\infty} E_k$, where $E_0 = [0, 1]$, and

- for $k_0 \leq k \leq k_1, k_2 < k \leq k_3, \ldots$, E_k is obtained by deleting the middle $1/3$ of each interval in E_{k-1};
- for $k_1 < k \leq k_2, k_3 < k \leq k_4, \ldots$, E_k is obtained by deleting the middle $3/5$ of each interval in E_{k-1}.

We estimate the lower and upper box dimensions of E by estimating $N_\delta(E)$, the least number of closed intervals of length δ that can cover E. (i) If n is even, then E_{k_n} is made up of 2^{k_n} intervals of length

$$\delta_n = \left(\frac{1}{3}\right)^{k_1}\left(\frac{1}{5}\right)^{k_2-k_1}\cdots\left(\frac{1}{3}\right)^{k_{n-1}-k_{n-2}}\left(\frac{1}{5}\right)^{k_n-k_{n-1}} < \left(\frac{1}{5}\right)^{k_n-k_{n-1}}.$$

Taking these intervals as a cover

$$\begin{aligned}\underline{\dim}_{\mathrm{B}} E &\leq \underline{\lim}_{n\to\infty}\frac{\log N_{\delta_n}(E)}{-\log\delta_n} \leq \underline{\lim}_{n\to\infty}\frac{\log 2^{k_n}}{\log 5^{k_n-k_{n-1}}}\\ &= \underline{\lim}_{n\to\infty}\frac{k_n\log 2}{(k_n-k_{n-1})\log 5} = \underline{\lim}_{n\to\infty}\frac{10k_{n-1}\log 2}{9k_{n-1}\log 5} = \frac{10\log 2}{9\log 5}.\end{aligned}$$

(ii) If n is odd, then E_{k_n} is made up of 2^{k_n} intervals of length

$$\begin{aligned}\delta_n &= \left(\frac{1}{3}\right)^{k_1}\left(\frac{1}{5}\right)^{k_2-k_1}\cdots\left(\frac{1}{5}\right)^{k_{n-1}-k_{n-2}}\left(\frac{1}{3}\right)^{k_n-k_{n-1}}\\ &> \left(\frac{1}{5}\right)^{k_{n-1}}\left(\frac{1}{3}\right)^{k_n-k_{n-1}}.\end{aligned}$$

Any interval of length δ_n meets at most two of the intervals in E_{k_n} and so, since E has points in every interval in E_{k_n},

$$\begin{aligned}
\overline{\dim}_{\mathrm{B}} E &\geq \overline{\lim}_{n\to\infty} \frac{\log N_{\delta_n}(E)}{-\log \delta_n} \geq \overline{\lim}_{n\to\infty} \frac{\log(2^{k_n}/2)}{\log(5^{k_{n-1}} 3^{k_n - k_{n-1}})} \\
&= \overline{\lim}_{n\to\infty} \frac{k_n \log 2 - \log 2}{k_{n-1}\log 5 + (k_n - k_{n-1})\log 3} \\
&= \overline{\lim}_{n\to\infty} \frac{10 k_{n-1}\log 2 - \log 2}{k_{n-1}\log 5 + 9k_{n-1}\log 3} \\
&= \frac{10\log 2}{\log 5 + 9\log 3} \geq \frac{10\log 2}{11\log 3}.
\end{aligned}$$

Since

$$\frac{10\log 2}{9\log 5} < \frac{10\log 2}{11\log 3}$$

$$\underline{\dim}_{\mathrm{B}} E < \overline{\dim}_{\mathrm{B}} E,$$

as required.

3.9 By monotonicity, $\overline{\dim}_{\mathrm{B}}(E \cup F) \geq \max\{\overline{\dim}_{\mathrm{B}} E, \overline{\dim}_{\mathrm{B}} F\}$.

Let $N_\delta(F)$ denote the least number of intervals of length δ that can cover a set F. Then $N_\delta(E\cup F) \leq N_\delta(E) + N_\delta(F) \leq 2\max\{N_\delta(E), N_\delta(F)\}$, so

$$\begin{aligned}
\overline{\dim}_{\mathrm{B}}(E\cup F) &= \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(E\cup F)}{-\log\delta} \\
&\leq \overline{\lim}_{\delta\to 0} \frac{\log(2\max\{N_\delta(E), N_\delta(F)\})}{-\log\delta} \\
&\leq \overline{\lim}_{\delta\to 0} \frac{\log 2}{-\log\delta} + \overline{\lim}_{\delta\to 0} \max\left\{\frac{\log N_\delta(E)}{-\log\delta}, \frac{\log N_\delta(F)}{-\log\delta}\right\} \\
&\leq 0 + \max\left\{\overline{\lim}_{\delta\to 0} \frac{\log N_\delta(E)}{-\log\delta}, \overline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log\delta}\right\} \\
&= \max\{\overline{\dim}_{\mathrm{B}}(E), \overline{\dim}_{\mathrm{B}}(F)\}.
\end{aligned}$$

Note that we cannot interchange 'max' and '$\underline{\lim}$' in the same way, so the argument fails for lower box dimensions.

3.10 The idea is to construct sets E and F such that at every scale one of E or F looks 'large' and the other looks 'small'. Let E be the set described in the Solution to Exercise 3.8. We construct a set F in a similar way, except that the scaling of intervals is complementary and the set is positioned to be disjoint from E. Thus set $k_n = 10^n$, for $n = 0, 1, 2, \ldots$ and let $F = \bigcap_{k=0}^{\infty} F_k$, where $F_0 = [2, 3]$, and

- for $k_0 \le k \le k_1$, $k_2 < k \le k_3, \ldots$, F_k is obtained by deleting the middle $3/5$ of each interval in F_{k-1};
- for $k_1 < k \le k_2$, $k_3 < k \le k_4, \ldots$, F_k is obtained by deleting the middle $1/3$ of each interval in F_{k-1}.

As in the solution to Exercise 3.8 we get that

$$\underline{\dim}_B E, \underline{\dim}_B F \le \frac{10 \log 2}{9 \log 5}.$$

For each $k = 1, 2, \ldots$, let δ_k denote the length of the longest intervals in $E_k \cup F_k$: there are 2^k such intervals, each of which meets $E \cup F$. Since any other interval of length δ_k meets at most two of these intervals, it follows that the smallest number of closed intervals of length δ_k that cover $E \cup F$ satisfies $N_{\delta_k}(E \cup F) \ge 2^k/2$. Now $\delta_k \ge \left(\frac{1}{3}\right)^{k/2} \left(\frac{1}{5}\right)^{k/2}$ and $\delta_k \ge (1/5)\delta_{k-1}$, so by the note after Definitions 3.1

$$\begin{aligned}
\underline{\dim}_B E \cup F &= \underline{\lim}_{\delta_k \to 0} \frac{\log N_{\delta_k}(E \cup F)}{-\log \delta_k} \ge \underline{\lim}_{k\to\infty} \frac{\log 2^k/2}{\log 5^{k/2} \log 3^{k/2}} \\
&= \underline{\lim}_{k\to\infty} \frac{k \log 2 - \log 2}{(k/2)\log 5 + (k/2)\log 3} = \frac{2\log 2}{\log 5 + \log 3} > \frac{10 \log 2}{9 \log 5}.
\end{aligned}$$

3.11 Since F is a countable set, $\dim_H F = 0$.

The box dimension calculation is similar to Example 3.5. Let $N_\delta(F)$ be the smallest number of sets of diameter at most δ that cover F. If $|U| = \delta < 1/2$ and k is the integer satisfying

$$\frac{2k-1}{k^2(k-1)^2} = \frac{1}{(k-1)^2} - \frac{1}{k^2} > \delta \ge \frac{1}{k^2} - \frac{1}{(k+1)^2} = \frac{2k+1}{(k+1)^2 k^2},$$

then U can cover at most one of the points $\left\{1, \frac{1}{4}, \ldots, \frac{1}{k^2}\right\}$. Thus $N_\delta(F) \ge k$ and hence

$$\begin{aligned}
\underline{\dim}_B F &= \underline{\lim}_{\delta\to 0} \frac{\log N_\delta(F)}{-\log \delta} \ge \underline{\lim}_{k\to\infty} \frac{\log k}{\log \frac{(k+1)^2 k^2}{2k+1}} \\
&= \underline{\lim}_{k\to\infty} \frac{\log k}{2\log(k+1) + 2\log k - \log 2 - \log(k+1/2)} = \frac{1}{3}.
\end{aligned}$$

On the other hand, if

$$\frac{1}{k(k-1)^2} > \delta \geq \frac{1}{(k+1)k^2},$$

then $k+1$ intervals of length δ cover $[0, 1/k^2]$ leaving $k-1$ points of F which can be covered by $k-1$ intervals of length δ. Thus

$$\begin{aligned}\overline{\dim}_{\mathrm{B}} F &= \overline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta} \leq \overline{\lim}_{k\to\infty}\frac{\log 2k}{\log k(k-1)^2} \\ &= \overline{\lim}_{k\to\infty}\frac{\log k + \log 2}{\log k + 2\log(k-1)} = \frac{1}{3}.\end{aligned}$$

Thus $\dim_{\mathrm{B}} F = 1/3$.

3.12 Let $E = [0,1]\cap\mathbb{Q}$ and $F = \{x \in [0,1] : x - \sqrt{2} \in \mathbb{Q}\}$, so that E and F are disjoint dense subsets of $[0, 1]$. If $\{B_i\}$ is any collection of disjoint balls (i.e. intervals) with centres in E and radii at most δ, then, by considering the lengths of the B_i, we see that $\sum_i |B_i| \leq 1 + \delta$. Moreover, taking B_i as nearly abutting intervals of lengths 2δ we can get $\sum_i |B_i| \geq 1$. Thus, since

$$\mathcal{P}^1_\delta(E) = \sup\left\{\sum_i |B_i| : \{B_i\}\text{are disjoint balls of radii } \leq \delta \text{ with centres in } F\right\},$$

we get $1 \leq \mathcal{P}^1_\delta(E) \leq 1+\delta$. Letting $\delta \to 0$ gives $\mathcal{P}^1_0(E) = 1$. In a similar way, $\mathcal{P}^1_0(E) = 1$ and $\mathcal{P}^1_0(E\cup F) = 1$. In particular $\mathcal{P}^1_0(E\cup F) \neq \mathcal{P}^1_0(E) + \mathcal{P}^1_0(F)$.

3.13 The von Koch curve F has (upper and lower) box dimensions equal to $\log 4/\log 3$. Moreover, by virtue of the self-similarity of F, $\dim_{\mathrm{B}}(F \cap V) = \log 4/\log 3$ for every open set V that intersects F. By Corollary 3.9, $\dim_{\mathrm{P}} F = \dim_{\mathrm{B}} F = \log 4/\log 3$.

3.14 Recall that the divider dimension of a curve C is defined as $\lim_{\delta\to 0}\log M_\delta(C)/ -\log\delta$ (assuming that this limit exists), where $M_\delta(C)$ is the maximum number of points $x_0, x_1, \ldots, x_m$ on C, in that order, with $|x_i - x_{i-1}| \geq \delta$ for $i = 1, 2, \ldots, m$.

By inspection of the von Koch curve C, taken to have of baselength 1, (see Figure 0.2), we have that if k is the integer such that $3^{-k-1} \leq \delta < 3^{-k}$, then $4^k < 4^k + 1 \le M_\delta(C) \le 4^{k+1} + 1 < 4^{k+2}$. Then

$$\frac{k\log 4}{(k+1)\log 3} = \frac{\log(4^k)}{-\log(3^{-k-1})} \le \frac{\log M_\delta(C)}{-\log\delta} \le \frac{\log(4^{k+2})}{-\log(3^{-k})}\frac{(k+2)\log 4}{\log 3}.$$

As $\delta \to 0$, $k \to \infty$, so taking limits gives that

$$\text{divider dimension } = \lim_{\delta\to 0}\frac{\log M_\delta(C)}{-\log\delta} = \frac{\log 4}{-\log 3}$$

(which, of course equals the Hausdorff and box dimensions of C).

3.15 Recall that the divider dimension of a curve C is defined as $\lim_{\delta\to 0}\log M_\delta(C)/-\log\delta$ (assuming that this limit exists), where $M_\delta(C)$ is the maximum number of points $x_0, x_1, \dots, x_m$ on C, in that order, with $|x_i - x_{i-1}| \ge \delta$ for $i = 1, 2, \dots, m$.

Consider Equivalent definition 3.1(v) of box dimension, taking $N_\delta(C)$ to be the greatest number of disjoint balls of radius δ with centres on C. Then if $B_1, \dots, B_{N_\delta(C)}$ is a maximal collection of disjoint balls of radii δ with centres on C, every ball B_i must contain at least one point x_j in any maximal sequence of points $x_0, x_1, \dots, x_1$ for the divider dimension, otherwise the centre of B_i may be added to the sequence to increase its length. Thus $N_\delta(C) \le M_\delta(C)$, so

$$\frac{N_\delta(C)}{-\log\delta} \le \frac{\log M_\delta(C)}{-\log\delta},$$

and taking limits as $\delta \to 0$ gives that the box dimension is less than or equal to the divider dimension, assuming both exist. (If not a similar inequality holds for lower and upper box and lower and upper divider dimensions.)

3.16 The middle λ Cantor set F may be constructed from the unit interval by removing 2^k open intervals of lengths $\lambda\left(\frac{1}{2}(1-\lambda)\right)^k$ for $k = 0, 1, 2, \dots$. Thus, denoting these complementary intervals by I_i, we have

$$\sum_i |I_i|^s = \sum_{k=0}^{\infty} 2^k\lambda^s\left(\frac{1}{2}(1-\lambda)\right)^{ks}.$$

This is a geometric series which converges if and only if the common ratio $2\left(\frac{1}{2}(1-\lambda)\right)^s < 1$, that is if $s > \log 2/\log(2/(1-\lambda))$, a number equal to the Hausdorff and box dimensions of F, see Exercise 2.14.

3.17 If $F_1 \subset F_2$ then any δ-cover of F_2 by rectangles is also a δ-cover of F_1, so that from the definition, $\mathcal{H}^{s,t}_\delta(F_1) \le \mathcal{H}^{s,t}_\delta(F_2)$, and letting $\delta \to 0$ gives $\mathcal{H}^{s,t}(F_1) \le \mathcal{H}^{s,t}(F_2)$. In particular, if $(s,t) \in \text{print}F_1$ then $0 < \mathcal{H}^{s,t}(F_1)$ so $0 < \mathcal{H}^{s,t}(F_2)$ giving $(s,t) \in \text{print}F_2$. Thus $\text{print}F_1 \subset \text{print}F_2$.

It follows at once that $\text{print}F_k \subset \text{print}\left(\bigcup_{i=1}^{\infty} F_i\right)$ for all k, so that $\bigcup_{i=1}^{\infty} \text{print}F_i \subset \text{print}\left(\bigcup_{i=1}^{\infty} F_i\right)$.

Now suppose $(s,t) \notin \text{print}F_i$ for all i. Then $\mathcal{H}^{s,t}(F_i) = 0$ for all i, so $\mathcal{H}^{s,t}\left(\bigcup_{i=1}^{\infty} F_i\right) = 0$ since $\mathcal{H}^{s,t}$ is a measure. We conclude that $(s,t) \notin \text{print}\left(\bigcup_{i=1}^{\infty} F_i\right)$. Thus $\bigcup_{i=1}^{\infty} \text{print}F_i = \text{print}\left(\bigcup_{i=1}^{\infty} F_i\right)$.

Suppose now that $s' + t' \le s + t$ and $t' \le t$. For any δ-cover of a set F by rectangles U_i with sides $a(U_i) \ge b(U_i)$, we have

$$\begin{aligned}\sum_i a(U_i)^s b(U_i)^t &\le \sum_i a(U_i)^{s-s'} a(U_i)^{s'} b(U_i)^{t-t'} b(U_i)^{t'}\\ &\le \sum_i a(U_i)^{s'} b(U_i)^{t'} a(U_i)^{s-s'+t-t'}\\ &\le \delta^{(s+t)-(s'+t')} \sum_i a(U_i)^{s'} b(U_i)^{t'}.\end{aligned}$$

It follows from the definition that if $0 < \delta < 1$ then $\mathcal{H}_\delta^{s,t}(F) \le \mathcal{H}_\delta^{s',t'}(F)$, so $\mathcal{H}^{s,t}(F) \le \mathcal{H}^{s',t'}(F)$. Thus if $(s,t) \in \text{print}F$ then $0 < \mathcal{H}^{s,t}(F) \le \mathcal{H}^{s',t'}(F)$, so $(s',t') \in \text{print}F$.

Since $\text{print}(F_1 \cup F_2) = \text{print}F_1 \cup \text{print}F_2$, taking F_1 and F_2 such that the union of their dimension prints is not convex will give a set $F_1 \cup F_2$ with non-convex dimension print. Taking F_1 a circle and F_2 the product of uniform Cantor sets of dimensions $\frac{1}{3}$ and $\frac{3}{4}$, will achieve this, see figure 3.3.

Chapter 4

4.1 We begin by noting that the Cantor tartan is equal to $(F \times \mathbb{R}) \cup (\mathbb{R} \times F)$. In Example 4.3 it is shown that $\dim_H(F \times [0,1]) = 1 + \log 2/\log 3$. Now $F \times [n, n+1]$ is a translate of $F \times [0,1]$, so for each integer n, $\dim_H(F \times [n, n+1]) = 1 + \log 2/\log 3$. Thus, by countable stability, $\dim_H(F \times \mathbb{R}) = 1 + \log 2/\log 3$. As $\mathbb{R} \times F$ is congruent to $F \times \mathbb{R}$ under a 90 degree rotation, $\dim_H(\mathbb{R} \times F) = 1 + \log 2/\log 3$. Finally,

$$\begin{aligned}&\dim_H((F \times \mathbb{R}) \cup (\mathbb{R} \times F))\\ &\quad = \max\{\dim_H(F \times \mathbb{R}), \dim_H(\mathbb{R} \times F)\} = 1 + \frac{\log 2}{\log 3}.\end{aligned}$$

4.2 Let F be the set of numbers in $[0,1]$ containing only even digits. Writing E_k for the set of numbers in $[0,1]$ containing only even digits in the first k decimal places, we have that $F = \bigcap_{k=0}^{\infty} E_k$. For each positive integer k, the 5^k intervals in E_k of length 10^{-k} form a 10^{-k}-cover of F and so

$\mathcal{H}^{\log 5/\log 10}_{10^{-k}}(F) \leq 5^k(10^{-k})^{\log 5/\log 10} = 5^k 5^{-k} = 1$. Letting $k \to \infty$ gives $\mathcal{H}^{\log 5/\log 10}(F) \leq 1$.

Now let μ be the natural mass distribution on F obtained by repeated subdivision of mass into 5 equal parts, so that each of the 5^k intervals of length 10^{-k} in E_k carries a mass of 5^{-k}. If $10^{-(k+1)} \leq |U| < 10^{-k}$ for some $k \geq 1$, then U can intersect at most one of the intervals in E_k and so

$$\mu(U) \leq 5^{-k} = (10^{-k})^{\log 5/\log 10} \leq (10|U|)^{\log 5/\log 10} = 5|U|^{\log 5/\log 10}.$$

It follows from the Mass distribution principle 4.2 that $\mathcal{H}^{\log 5/\log 10}(F) \geq 1/5$. Combining these results, we see that $\mathcal{H}^{\log 5/\log 10}(F)$ is positive and finite, so that $\dim_{\mathrm{H}} F = \log 5/\log 10$.

4.3 Let F be the Cantor dust depicted in figure 0.4. For each positive integer k, the $n_k = 4^k$ squares of diameter $\delta_k = 4^{-k}\sqrt{2}$ in E_k form a cover of F and so it follows from Proposition 4.1 that

$$\dim_{\mathrm{H}} F \leq \underline{\lim}_{k\to\infty} \frac{\log 4^k}{-\log(4^{-k}\sqrt{2})} = \underline{\lim}_{k\to\infty} \frac{k \log 4}{k\log 4 - \log\sqrt{2}} = 1.$$

Now let μ be the natural mass distribution on F, so that each of the 4^k squares of side 4^{-k} in E_k carries a mass of 4^{-k}. If $4^{-(k+1)} \leq |U| < 4^{-k}$ for some $k \geq 1$, then U can intersect at most one of the squares in E_k and so

$$\mu(U) \leq 4^{-k} \leq 4|U|.$$

It follows from the Mass distribution principle 4.2 that $\dim_{\mathrm{H}} F \geq 1$. Combining these results, we deduce that $\dim_{\mathrm{H}} F = 1$ as required.

4.4 If $\lambda = \frac{1}{2}$ then $F = [0, 1]$ so $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = 1$.

Thus assume $0 < \lambda < \frac{1}{2}$. It is easy to see that F is a subset of the interval $[0, \lambda/(1-\lambda)]$ and moreover F is the union of two similar copies of itself at scale λ, that is $F = (F \cap [0, \lambda^2/(1-\lambda)]) \cup (F \cap [\lambda, \lambda/(1-\lambda)])$. Thus F may be constructed by a Cantor-type construction, repeatedly replacing intervals by a pair of subintervals each of length λ times that of the parent interval. Let E_k be the set of 2^k intervals of lengths $\lambda^{k+1}/(1-\lambda)$ obtained at the kth stage of this construction. Then $F = \bigcap_{k=0}^{\infty} E_k$. For each positive integer k, the 2^k intervals in E_k form a cover of F and so it follows from Proposition 4.1 that

$$\begin{aligned}\overline{\dim}_{\mathrm{B}} F &\leq \overline{\lim_{k\to\infty}} \frac{\log 2^k}{-\log \lambda^{k+1}/(1-\lambda)} \\ &= \overline{\lim_{k\to\infty}} \frac{k\log 2}{-(1+k)\log\lambda + \log(1-\lambda)} = \frac{\log 2}{-\log\lambda}.\end{aligned}$$

Now let μ be the natural mass distribution on F so that each of the 2^k intervals of lengths $\lambda^{k+1}/(1-\lambda)$ in E_k carries a mass of 2^{-k}. If $\lambda^{k+1}(1-2\lambda)/(1-\lambda) \leq |U| < \lambda^k(1-2\lambda)/(1-\lambda)$ for some $k \geq 1$, then U can intersect at most one of the intervals in E_k and so

$$\mu(U) \leq 2^{-k} \leq (\lambda^k)^{-\log 2/\log\lambda} \leq \left(\frac{1-\lambda}{\lambda(1-2\lambda)}|U|\right)^{-\log 2/\log\lambda}$$
$$= c|U|^{-\log 2/\log\lambda}.$$

where $c > 0$ is independent of U. It follows from the Mass distribution principle 4.2 that $\dim_{\mathrm{H}} F \geq -\log 2/\log\lambda$. Combining these results, we deduce that $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = -\log 2/\log\lambda$.

4.5 Let E'_k be the set of 2^k intervals of lengths 3^{-k} obtained at the kth stage of the usual construction of the middle third Cantor set. Then $E_k = E'_k \times E'_k$ consists of 4^k squares of sides 3^{-k} and diameters $3^{-k}\sqrt{2}$, and $F \times F = \bigcap_{k=0}^{\infty} E_k$. For each positive integer k, the 4^k squares in E_k form a $3^{-k}\sqrt{2}$-cover of F and so $\mathcal{H}^{\log 4/\log 3}_{3^{-k}\sqrt{2}}(F) \leq 4^k(3^{-k}\sqrt{2})^{\log 4/\log 3} = 2^{\log 2/\log 3}4^k4^{-k} = 2^{\log 2/\log 3}$. Letting $k \to \infty$ gives $\mathcal{H}^{\log 4/\log 3}(F) \leq 2^{\log 2/\log 3}$.

Now let μ be the natural mass distribution on $F \times F$ obtained by repeated subdivision of mass into 4 equal parts, so that each of the 4^k squares in E_k carries a mass of 4^{-k}. If $3^{-(k+1)} \leq |U| < 3^{-k}$ for some $k \geq 1$, then U intersects at most one of the squares in E_k and so

$$\mu(U) \leq 4^{-k} = (3^{-k})^{\log 4/\log 3} \leq (3|U|)^{\log 4/\log 3} = 4|U|^{\log 4/\log 3}$$

It follows from the Mass distribution principle 4.2 that $\mathcal{H}^{\log 4/\log 3}(F) \geq 1/4$. Combining these results, we see that $\mathcal{H}^{\log 4/\log 3}(F)$ is positive and finite, so that $\dim_{\mathrm{H}} F = \log 4/\log 3$.

4.6 With F the middle third Cantor set, it is easily checked that

$$F_0 \equiv \left(F \cap \left[\tfrac{2}{3}, 1\right]\right) \times \left[0, \tfrac{1}{3}\right] \subset \left\{(x, y) \in \mathbb{R}^2 : x \in F \text{and } 0 \leq y \leq x^2\right\} \subset$$
$$F \times [0, 1] \equiv F_1.$$

We showed in Example 4.3 that $\dim_{\mathrm{H}} F_1 = 1 + \log 2/\log 3$, and F_0 is a similar copy of F_1 at scale $\frac{1}{3}$, so $\dim_{\mathrm{H}} F_0 = 1 + \log 2/\log 3$. It is immediate that the dimension of the intermediate set is also $1 + \log 2/\log 3$.

4.7 Let F be the set described in Example 4.5. For each positive integer k, the $n_k = m^k$ intervals of length $\delta_k = \lambda^k$ in E_k form a cover of F and so

Proposition 4.1 gives that

$$\dim_{\mathrm{H}} F \le \overline{\dim}_{\mathrm{B}} F \le \overline{\lim}_{k\to\infty} \frac{\log m^k}{-\log \lambda^k} = \frac{\log m}{-\log \lambda}.$$

Since $n_k \delta_k^{\log m/(-\log\lambda)} = m^k \lambda^{k \log m/(-\log\lambda)} = 1$, it also follows from Proposition 4.1 that $\mathcal{H}^{\log m/(-\log\lambda)}(F) < \infty$.

Now let μ be the natural mass distribution on F so that each of the m^k intervals of length λ^k in E_k carries a mass of m^{-k}. If $\lambda^{(k+1)} \le |U| < \lambda^k$ for some $k \ge 1$, then U can intersect at most two of the intervals in E_k, so

$$\mu(U) \le 2m^{-k} = 2mm^{-(k+1)} = 2m\lambda^{(k+1)\log m/-\log\lambda} \le 2m|U|^{\log m/-\log\lambda}.$$

It follows from the Mass distribution principle 4.2 that $\mathcal{H}^{\log m/-\log\lambda}(F) \ge \mu(F)/2m = 1/(2m) > 0$ and $\dim_{\mathrm{H}} F \ge \log m/-\log\lambda$. Combining these results, we deduce that $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = \log m/-\log\lambda$ and $0 < \mathcal{H}^{\log m/-\log\lambda}(F) < \infty$.

4.8 We show inductively that there are integers m and $a_2, a_3, \ldots$ with $0 \le a_j \le j-1$, such that for all $k = 1, 2, \ldots$, the number x may be expressed in the form

$$x = m + \frac{a_2}{2!} + \cdots + \frac{a_k}{k!} + y_k \qquad (*)$$

where $0 \le y_k < 1/k!$. This is clear when $k = 1$ (expressing x as in integer plus a fractional part), so suppose inductively that $(*)$ holds for some $k \ge 1$. Since $0 \le k!y_k < 1$, we may write $k!y_k = a_{k+1}/(k+1) + z_{k+1}$ where a_{k+1} is an integer and $0 \le z_{k+1} < 1/(k+1)$. Thus $y_k = a_{k+1}/(k+1)! + y_{k+1}$, where $0 \le y_{k+1} = z_{k+1}/k! < 1/(k+1)!$. Substituting into $(*)$ gives the same formula with k replaced by $k+1$, completing the inductive step.

Letting $k \to \infty$ in $(*)$ gives

$$x = m + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots,$$

convergence of the series following from comparison with the exponential series.

To find the dimension of

$$F = \{x = m + \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots : m = 0 \text{ and } a_k \text{ is even for } k = 2, 3, \ldots\}$$

we use the result of Example 4.6. Writing

$$E_k = \{x = \frac{a_2}{2!} + \frac{a_3}{3!} + \cdots : a_2, \ldots, a_k \text{ are even}\},$$

we have $F = \bigcap_{k=1}^{\infty} E_k$ as in the general construction (4.3). Each interval of E_{k-1} contains $m_k = \lceil \frac{1}{2}k \rceil$ intervals of E_k, with each interval of E_k of length $1/k!$ and separated by at least $\epsilon_k = 1/k!$. The formula of Example 4.6 gives

$$\begin{aligned}\dim_{\mathrm{H}} F &\geq \underline{\lim}_{k\to\infty} \frac{\log m_1 \ldots m_{k-1}}{-\log(m_k \epsilon_k)} = \underline{\lim}_{k\to\infty} \frac{\log 2.2.3.3 \ldots \lceil \frac{1}{2}(k-1) \rceil}{-\log(\lceil \frac{1}{2}k \rceil / k!)} \\ &\geq \underline{\lim}_{k\to\infty} \frac{\log(\lceil \frac{1}{2}(k-2) \rceil !^2)}{\log k! - \log \frac{1}{2}k} = \underline{\lim}_{k\to\infty} \frac{2(\frac{1}{4}k \log k)}{\frac{1}{2}k \log k - \log \frac{1}{2}k} = 1\end{aligned}$$

where we have used Stirling's formula in the form $\log n! \sim \frac{1}{2}\log 2\pi + \frac{1}{2}(n + \frac{1}{2})\log n - n$.

We conclude that $\dim_{\mathrm{H}} F = 1$.

4.9 An easy way to do this is as follows. For each $0 < s < 1$, there is a compact set $E_s \subset [0, 1]$ such that $\dim_{\mathrm{H}} E_s = s$. (For example, the 'middle λ Cantor set', see after Example 4.5, has Hausdorff dimension $\log 2/\log(2/(1-\lambda))$ for $0 < \lambda < 1$, so taking $\lambda = 1 - 2^{1-1/s}$ gives a suitable set E_s.) For $n = 1, 2, \ldots$ let F_n be a similar copy of $E_{1-1/n}$, scaled and translated so $F_n \subset [1/n, 1/(n+1)]$. Then $\dim_{\mathrm{H}} F_n = 1 - 1/n$ and $\mathcal{H}^1(F_n) = 0$. Set $F = \{0\} \cup \bigcup_{n=1}^{\infty} F_n$. Then F is compact, $\dim_{\mathrm{H}} F = \sup_{1 \leq n < \infty} \dim_{\mathrm{H}} F_n = 1$, and $\mathcal{H}^1(F) = \sum_{n=1}^{\infty} \mathcal{H}^1(F_n) = 0$, as required.

4.10 First we show that if $\mathcal{H}^s(F) = \infty$ then for every c with $0 < c < \infty$ there is a Borel $E \subset F$ such that $c < \mathcal{H}^s(E) < \infty$. For suppose to the contrary. Then $a \equiv \sup\{\mathcal{H}^s(E) : E$ is a Borel subset of F with $\mathcal{H}^s(E) < \infty\} < \infty$. There is a Borel set E with $\mathcal{H}^s(E) = a$ (for if we take a sequence of Borel sets E_k with $\mathcal{H}^s(E_k) \nearrow a$ then $\mathcal{H}^s\left(\bigcup_{k=1}^{\infty} E_k\right) = a$). Thus $F \setminus E$ is a Borel set with $\mathcal{H}^s(F \setminus E) \geq \mathcal{H}^s(F) - \mathcal{H}^s(E) = \infty$. By Theorem 4.10, $F \setminus E$ has a Borel subset G with $\mathcal{H}^s(G) > 0$, so $E \cup G$ is a union of disjoint Borel sets, so is Borel, with $\infty > \mathcal{H}^s(E \cup G) = \mathcal{H}^s(E) + \mathcal{H}^s(G) > a$, a contradiction.

Thus, given F with $\mathcal{H}^s(F) = \infty$ and $0 < c < \infty$ let E_0 be a Borel subset of F such that $c < \mathcal{H}^s(E_0) < \infty$. Since $E_0 = E_0 \cap \bigcup_{n=1}^{\infty}[-n, n]$ there is an integer n such that $c \leq \mathcal{H}^s(E_0 \cap [-n, n]) < \infty$. The function ϕ given by $\phi(x) = \mathcal{H}^s(E_0 \cap [-n, x])$ is continuous and increasing in x, by continuity of finite measures, see Exercise 1.18. Since $\phi(-n) = 0$ and $\phi(n) > c$, the intermediate value theorem gives that there is an x, $-n < x < n$ such that $\phi(x) = c$. Thus $E = E_0 \cap [-n, x]$ is a Borel subset of F with $\mathcal{H}^s(E) = c$, as required.

4.11 Consider the usual construction of the middle third Cantor set F, the kth stage of the approximation E_k comprising 2^k intervals of lengths 3^{-k}. Let

μ be the natural mass distribution on F, so that each interval I of E_k has $\mu(I) = 2^{-k}$. For each distinct $x, y \in F$ there is an integer $k \geq 1$ such that x and y are in the same interval of E_{k-1} but with $x \in I$ and $y \in I'$ with I and I' distinct intervals of E_k. We decompose the energy integral into a sum of pairs of intervals of this type. Thus for $s > 0$

$$\begin{aligned}\iint \frac{d\mu(x)d\mu(y)}{|x-y|^s} &\leq \sum_{k=1}^{\infty} \sum_{I \neq I'; I, I' \in E_k} \int_{x \in I} \int_{y \in I'} \frac{d\mu(x)d\mu(y)}{|x-y|^s} \\ &\leq \sum_{k=1}^{\infty} 2^k \frac{2^{-k}2^{-k}}{3^{-ks}} \\ &= \sum_{k=1}^{\infty} \left(\frac{3^s}{2}\right)^k .\end{aligned}$$

This series converges if and only if $s < \log 2/\log 3$ (in which case the energy is at most $(3^s/2)/(1-(3^s/2))$, so by Theorem 4.13, $\dim_{\mathrm{H}} F \geq \log 2/\log 3$.

Chapter 5

5.1 Let F be a Borel subset of $\mathbb{R}^2$ with $0 < \mathcal{L}^2(F) < \infty$, so F is a 2-set. Assume for the time being that F is bounded, say $F \subset B$ for some disc B. Noting that $\mathcal{H}^2(A) = c\mathcal{L}^2(A)$ for a constant $c > 0$, (5.3) and Proposition 5.1(a) applied to F and then $B \setminus F$ gives

$$\lim_{r\to 0} \frac{\mathcal{L}^2(F \cap B(x,r))}{\mathcal{L}^2(B(x,r))} = 0$$

for almost all $x \notin F$ and

$$\begin{aligned}\lim_{r\to 0} \frac{\mathcal{L}^2(F \cap B(x,r))}{\mathcal{L}^2(B(x,r))} &= \lim_{r\to 0} \frac{\mathcal{L}^2(B \cap B(x,r))}{\mathcal{L}^2(B(x,r))} \\ &\quad - \lim_{r\to 0} \frac{\mathcal{L}^2((B \setminus F) \cap B(x,r))}{\mathcal{L}^2(B(x,r))} = 1 - 0\end{aligned}$$

for almost all $x \notin B \setminus F$ so for almost all $x \in F$. This is the Lebesgue density theorem for bounded F. For unbounded F, we have the result for $F \cap B$ for every disc B, so since the Lebesgue density at x depends only on F in an neighbourhood of x, the result follows for all Borel sets F.

5.2 Let $x \in \mathbb{R}$ with $f'(x) = c$. Given $0 < \epsilon < c$, there exists $\delta > 0$ such that if $|y - x| < \delta$, then $c - \epsilon < f'(y) < c + \epsilon$, by continuity of f'. By the mean value theorem

$$|f(y) - f(z)| = |f'(w)||y - z|$$

for $y, z \in B(x, \delta)$, for some $w \in B(x, \delta)$, so f is bi-Lipschitz on $B(x, \delta)$ with

$$(c - \epsilon)|z - y| \le |f(y) - f(z)| \le (c + \epsilon)|y - z|.$$

Thus if $0 < r < \delta$, we have

$$B(f(x), (c - \epsilon)r) \subset f(B(x, r)) \subset B(f(x), (c + \epsilon)r)$$

and by (2.9)

$$\begin{aligned}(c - \epsilon)^s \mathcal{H}^s(F \cap B(x, r)) &\le \mathcal{H}^s(f(F \cap B(x, r))) \\ &\le (c + \epsilon)^s \mathcal{H}^s(F \cap B(x, r)).\end{aligned}$$

Hence

$$\begin{aligned}\frac{\mathcal{H}^s(f(F) \cap B(f(x), (c - \epsilon)r))}{(c + \epsilon)^s} &\le \mathcal{H}^s(F \cap B(x, r)) \\ &\le \frac{\mathcal{H}^s(f(F) \cap B(f(x), (c + \epsilon)r))}{(c - \epsilon)^s}\end{aligned}$$

so

$$\left(\frac{c - \epsilon}{c + \epsilon}\right)^s \frac{\mathcal{H}^s(f(F) \cap B(f(x), (c - \epsilon)r))}{(2(c - \epsilon)r)^s} \le \frac{\mathcal{H}^s(F \cap B(x, r))}{(2r)^s}$$

$$\le \left(\frac{c + \epsilon}{c - \epsilon}\right)^{-s} \frac{\mathcal{H}^s(f(F) \cap B(f(x), (c + \epsilon)r))}{(2(c + \epsilon)r)^s}.$$

Taking limits as $r \searrow 0$ gives

$$\left(\frac{c - \epsilon}{c + \epsilon}\right)^s \underline{D}^s(f(F), f(x)) \le \underline{D}^s(F, x) \le \left(\frac{c + \epsilon}{c - \epsilon}\right)^s \underline{D}^s(f(F), f(x)).$$

This is true for all $\epsilon > 0$, so $\underline{D}^s(f(F), f(x)) = \underline{D}^s(F, x)$. Similarly, taking upper limits gives $\overline{D}^s(f(F), f(x)) = \overline{D}^s(F, x)$.

5.3 Let F be the middle third Cantor set. If $x \notin F$ then, since F is closed, $F \cap B(x,r) = \emptyset$ for sufficiently small r, so $\underline{D}^s(F,x) = \overline{D}^s(F,x) = 0$.

Let $x \in F$. For $k = 1, 2, \ldots$, the (interior of the) interval $B(x, 3^{-k})$ intersects just one kth level interval in E_k in the usual construction of the Cantor set, see figure 0.1. Thus, with $s = \log 2/\log 3$, $\mathcal{H}^s(F \cap B(x, 3^{-k})) \leq 2^{-k}$ (the $\mathcal{H}^s$-measure of a kth level interval), so

$$\frac{\mathcal{H}^s(F \cap B(x, 3^{-k}))}{(2.3^{-k})^s} \leq \frac{2^{-k}}{2^s 2^{-k}} = 2^{-s}.$$

It follows that

$$\begin{aligned}\underline{D}^s(F,x) &= \underline{\lim}_{r\to 0}\frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s}\\ &\leq \underline{\lim}_{k\to\infty}\frac{\mathcal{H}^s(F \cap B(x, 3^{-k}))}{(2.3^{-k})^s} = 2^{-s}.\end{aligned}$$

Since $\underline{D}^s(F,x) < 1$ for all x, F is irregular.

5.4 Let F be the dust of figure 5.4, so at the kth stage of construction, E_k consists of 4^k squares of sides 4^{-k} each with $\mathcal{H}^1$-measure 4^{-k}. For $k = 1, 2, \ldots$, and $x \in F$, we have $\mathcal{H}^1(F \cap B(x, 4^{-k})) \leq 4^{-k}$, since $B(x, 4^{-k})$ intersects just one square of E_k. Thus

$$\begin{aligned}\underline{D}^1(F,x) &= \underline{\lim}_{r\to 0}\frac{\mathcal{H}^1(F \cap B(x,r))}{2r} \leq \underline{\lim}_{k\to\infty}\frac{\mathcal{H}^1(F \cap B(x, 4^{-k}))}{2.4^{-k}}\\ &\leq \lim_{k\to\infty}\frac{4^{-k}}{2.4^{-k}} \leq \frac{1}{2}.\end{aligned}$$

In particular, F is irregular.

Similarly, if $x \in F$ then $\mathcal{H}^1(F \cap B(x, 4^{-k}\sqrt{2})) \geq 4^{-k}$, since $B(x, 4^{-k}\sqrt{2})$ contains a complete square of E_k. Thus

$$\begin{aligned}\overline{D}^1(F,x) &= \overline{\lim}_{r\to 0}\frac{\mathcal{H}^1(F \cap B(x,r))}{2r} \geq \overline{\lim}_{k\to\infty}\frac{\mathcal{H}^1(F \cap B(x, 4^{-k}\sqrt{2}))}{2.4^{-k}\sqrt{2}}\\ &\geq \lim_{k\to\infty}\frac{4^{-k}}{2.4^{-k}\sqrt{2}} \geq \frac{1}{2\sqrt{2}}.\end{aligned}$$

In fact for almost all $x \in F$, the point x lies arbitrarily close to the centre of squares in E_k for large k, so that, given $\epsilon > 0$, $B(x, 4^{-k}(\frac{1}{2}\sqrt{2} + \epsilon))$ contains a complete square of E_k for infinitely many k. Proceeding just as above gives that

$$\overline{D}^1(F,x) \geq \lim_{k\to\infty} \frac{4^{-k}}{4^{-k}(\sqrt{2}+2\epsilon)} = \frac{1}{\sqrt{2}+2\epsilon}.$$

Thus for $\mathcal{H}^1$-almost all $x \in F$, $\overline{D}^1(F,x) \geq 1/\sqrt{2}$.

5.5 Let F be an irregular s-set with $0 < s < 1$. Suppose for some $0 < d < 1$ there is a subset $F_1 \subset F$ with $\mathcal{H}^s(F_1) > 0$ such that $\underline{D}^s(F,x) > d$ for all $x \in F_1$. By Proposition 5.1(b) $\overline{D}^s(F,x) \leq 1$ for almost all $x \in F$. Given $\epsilon > 0$, Egoroff's theorem guarantees that there exists $r_0 > 0$ and a Borel set $E \subset F_1 \subset F$ with $\mathcal{H}^s(E) > 0$ and such that

$$d(1-\epsilon) \leq (2r)^{-s}\mathcal{H}^s(F \cap B(x,r)) \leq (1+\epsilon)$$

for all $x \in E$ and $r < r_0$.

Let y be a cluster point of E. Let η be a number with $0 < \eta < 1$ and let $A_{r,\eta}$ be the annulus $B(y, r(1+\eta)) \setminus B(y, r(1-\eta))$. Then for $r < \frac{1}{2}r_0$,

$$\begin{aligned}\frac{\mathcal{H}^s(F\cap A_{r,\eta})}{(2r)^s} &= \frac{\mathcal{H}^s(F\cap B(y,r(1+\eta)))}{(2r)^s} - \frac{\mathcal{H}^s(F\cap B(y,r(1-\eta)))}{(2r)^s}\\ &\leq (1+\epsilon)(1+\eta)^s - (1-\epsilon)d(1-\eta)^s.\end{aligned}$$

For a sequence of $r \searrow 0$ we may find $x \in E$ with $|x-y| = r$. For all $\epsilon > 0$, $B(x, r\eta(1-\epsilon)) \subset A_{r,\eta}$, so since $d(1-\epsilon) \leq (2r\eta(1-\epsilon))^{-s}$ $\mathcal{H}^s(B(x, r\eta(1-\epsilon)))$, we get

$$d(1-\epsilon)^{1+s}\eta^s \leq (1+\epsilon)(1+\eta)^s - (1-\epsilon)d(1-\eta)^s.$$

Since this is true for all $\epsilon > 0$ and $0 < \eta < 1$, we conclude that

$$d \leq \frac{(1+\eta)^s}{\eta^s + (1-\eta)^s}$$

for all $0 < \eta < 1$. We minimize this expression using elementary calculus. Differentiating, and equating to 0 gives

$$0 = s(\eta^s + (1-\eta)^s)(1+\eta)^{s-1} - s(1+\eta)^s(\eta^{s-1} - (1-\eta)^{s-1})$$

which simplifies to

$$\eta^{s-1} = 2(1-\eta)^{s-1}$$

so $\eta = 2^{1/(s-1)}/(1 + 2^{1/(s-1)})$ gives the minimum. Thus

$$d \leq \frac{(1+\eta)^s}{\eta^s + (1-\eta)^s} = \frac{(1+2.2^{1/(s-1)})^s}{2^{s/(s-1)}+1} = (1+2^{s/(s-1)})^{s-1}.$$

We conclude that $\underline{D}(F, x) \leq (1 + 2^{s/(s-1)})^{s-1}$ for almost all $x \in F$.

5.6 The simplest example is $F = [0, 1] \setminus \mathbb{Q}$, which has $\mathcal{L}^1(F) = 1$ (since $\mathbb{Q}$ is countable) and is totally disconnected, since there is a rational between any two distinct real numbers.

A more interesting example is a 'fat fractal' which may be obtained by a Cantor set construction, with $E_0 = [0, 1]$ and with E_k obtained from E_{k-1} by removing the middle proportion 2^{-k} from each of the 2^{k-1} intervals of E_{k-1}. As usual, $F = \cap_{k=0}^{\infty} E_k$. Clearly F is totally disconnected. Calculating the lengths of the intervals, and noting that $(1 - x) \geq \exp(-2x)$ for $0 < x < \frac{1}{2}$,

$$\mathcal{L}^1(E_k) = 2^k \times \frac{1}{2^k}\left(1-\frac{1}{2}\right)\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{2^k}\right)$$

$$\geq \exp(-1)\exp(-1/2)\cdots\exp(-1/2^{k-1}) \geq \exp(-2)$$

for all k. Thus $\mathcal{L}^1(F) \geq \exp(-2) > 0$.

5.7 Since $\mathcal{H}^s(E \cap B(x, r)) \leq \mathcal{H}^s(F \cap B(x, r))$ for all x and r, we have $\overline{D}^s(E, x) \leq \overline{D}^s(F, x)$ for all x. For $\mathcal{H}^s$-almost all $x \in E \setminus F$, we have $\overline{D}^s(F, x) = 0$ by Proposition 5.1(a), and so $\overline{D}^s(E, x) = 0$. On the other hand, for almost all $x \in E \setminus F$ we have $0 < \overline{D}^s(E \setminus F, x) \leq \overline{D}^s(E, x)$, by Proposition 5.1(b). We conclude $\mathcal{H}^s(E \setminus F) = 0$.

We cannot conclude $E \subset F$: for a counter example, take F to be an s-set where $s > 0$ and $E = F \cup \{x\}$ for some $x \notin F$.

5.8 The simplest approach is to use Theorem 5.9. If the F_k are all regular, each F_k may be covered by a countable union of rectifiable curves except for a set of $\mathcal{H}^1$-measure 0. Thus $\cup_{k=1}^{\infty} F_k$ may be covered by a countable union of rectifiable curves, except for a countable union of sets of $\mathcal{H}^1$-measure 0, a set which has $\mathcal{H}^1$-measure 0. (Recall that a countable union of countable sets is countable.) Since $\cup_{k=1}^{\infty} F_k$ in a 1-set, it fulfils the criteria of Theorem 5.9 to be regular.

Now suppose that the F_k are all irregular. If C is a rectifiable curve, $\mathcal{H}^1(C \cap F_k) = 0$, so

$$\mathcal{H}^1(C \cap \cup_{k=1}^{\infty} F_k) = \mathcal{H}^1(\cup_{k=1}^{\infty}(C \cap F_k)) \leq \Sigma_{k=1}^{\infty}\mathcal{H}^1(C \cap F_k) = 0.$$

By Theorem 5.9, $\cup_{k=1}^{\infty} F_k$ is irregular.

5.9 Suppose $\mathcal{H}^1(E \cap F) > 0$, so $E \cap F$ is a 1-set. The $E \cap F$ is a subset of a regular 1-set E so is regular, and also a subset of an irregular 1-set F so is irregular, by the remark after (5.4). Thus $E \cap F$ is both regular and irregular, so almost all points $x \in E \cap F$ are both regular and irregular. This is absurd, so we conclude that $\mathcal{H}^1(E \cap F) = 0$.

Chapter 6

6.1 (a) If $\frac{1}{2} < \lambda < 1$ then

$$1 > \frac{2\log 2}{\log(2/(1-\lambda))} = \dim_{\mathrm{H}}\mathrm{proj}_\theta E$$

for almost all θ, by Theorem 6.1. If $0 < \lambda \leq \frac{1}{2}$ then $\dim_{\mathrm{H}} E \geq 1$, so $\dim_{\mathrm{H}} \mathrm{proj}_\theta E = 1$ for almost all θ, by Theorem 6.1.

(b) We have $\mathrm{proj}_0 E = \mathrm{proj}_{\pi/2} E = F$, so

$$\dim_{\mathrm{H}}\mathrm{proj}_0 E = \dim_{\mathrm{H}}\mathrm{proj}_{\pi/2} E = \dim_{\mathrm{H}} F = \frac{\log 2}{\log(2/(1-\lambda))}.$$

6.2 With C as the unit circle in the complex plane, $f : [0,1] \to C$ given by $f(\phi) = e^{2\pi i\phi}$ is Lipschitz, since $|f(\phi_1) - f(\phi_2)| = |e^{2\pi i\phi_1} - e^{2\pi i\phi_2}| \leq 2\pi|\phi_1 - \phi_2|$. Thus $\dim_{\mathrm{H}} E = \dim_{\mathrm{H}} f(F) \leq \dim_{\mathrm{H}} F = \log 2/\log 3$, so $\dim_{\mathrm{H}} \mathrm{proj}_\theta E \leq \log 2/\log 3$ for all θ.

On the other hand, given θ, we may choose a basic interval I of the Cantor set such that the arc $A = \{f(\phi) : \phi \in I\}$ has all its tangents making angles at most $\psi < \frac{1}{2}\pi$ with the line L_θ in direction θ. Then $\mathrm{proj}_\theta : A \to L_\theta$ is bi-Lipschitz, so $\dim_{\mathrm{H}}\mathrm{proj}_\theta E \geq \dim_{\mathrm{H}}\mathrm{proj}_\theta(E \cap A) = \log 2/\log 3$. Hence $\dim_{\mathrm{H}}\mathrm{proj}_\theta E = \log 2/\log 3$ for all θ.

6.3 Let E be the middle λ Cantor set, with λ chosen so that $s = \log 2/\log(2/(1-\lambda)) = \dim_{\mathrm{H}} E$. Then E is an s-set. Let $F = \{(x,y) : (x = 0 \text{ and } y \in E) \text{ or } (x \in E \text{ and } y = 0)\}$. Then for all $\theta \neq 0, \frac{\pi}{2}$, the projection $\mathrm{proj}_\theta F$ is the union of two similar copies of E, so is an s-set. Also $\mathrm{proj}_0 E$ and $\mathrm{proj}_{\frac{\pi}{2}} E$ are congruent to E and so are s-sets.

6.4 We have by Theorem 6.1 that, for almost all θ,

$$\begin{aligned}\dim_{\mathrm{H}}\mathrm{proj}_\theta(E \times F) &= \min\{1, \dim_{\mathrm{H}}(E \times F)\} \geq \min\{1, \dim_{\mathrm{H}} E + \dim_{\mathrm{H}} F\}\\ &> \max\{\dim_{\mathrm{H}} E, \dim_{\mathrm{H}} F\},\end{aligned}$$

since $0 < \dim_{\mathrm{H}} E, \dim_{\mathrm{H}} F < 1$. But $\dim_{\mathrm{H}}\mathrm{proj}_0(E \times F) = \dim_{\mathrm{H}} E$ and $\dim_{\mathrm{H}} \mathrm{proj}_{\frac{\pi}{2}}(E \times F) = \dim_{\mathrm{H}} F$, so the projections onto the coordinate axes have exceptionally small dimensions in the sense of Theorem 6.1.

6.5 Assume without loss of generality that $\theta = 0$ and that F is bounded, say $F \subset [a, b] \times [a, b]$ (transforming by a congruence if necessary). Then $F \subset (\text{proj}_0 F) \times [a, b]$, so

$$\dim_{\text{H}} F \le \dim_{\text{H}}((\text{proj}_0 F) \times [a, b]) = \dim_{\text{H}}(\text{proj}_0 F) + 1$$

by a direct covering argument, or see Corollary 7.4. This formula extends to unbounded sets F in the usual way, expressing F as a countable union of bounded sets $F = \bigcup_{j=1}^{\infty}(F \cap ([-j, j] \times [-j, j]))$ and using countable stability of Hausdorff dimension $\dim_{\text{H}} F = \sup_{1 \le j < \infty} \dim_{\text{H}}(F \cap ([-j, j] \times [-j, j]))$.

6.6 Let $x \ne y \in F \subset \mathbb{R}^2$ where F is an irregular 1-set. By Theorem 6.4 we may choose a direction θ such that $\text{length}(\text{proj}_\theta F) = \mathcal{H}^1(\text{proj}_\theta F) = 0$ and such that $\text{proj}_\theta x \ne \text{proj}_\theta y$. Thus we may choose $z \in L_\theta$, the line through the origin in direction θ, with z between $\text{proj}_\theta x$ and $\text{proj}_\theta y$ such that $z \notin \text{proj}_\theta F$. Thus the line L through z and perpendicular to L_θ does not intersect F, so if U and V are the two open half-planes bounded by L, we have $F = (F \cap U) \cup (F \cap V)$ with $x \in F \cap U$ and $y \in F \cap V$. Thus x and y are in different connected components of F. This is true for all $x \ne y$, so F is totally disconnected.

6.7 Let $x \ne y \in F \subset \mathbb{R}^2$ and let θ be a direction other than that of the segment $[x, y]$, so $\text{proj}_\theta x \ne \text{proj}_\theta y$. If $\mathcal{L}^1(\text{proj}_\theta F) = 0$, then we may find $z \in L_\theta$, the line through the origin in direction θ, with z between $\text{proj}_\theta x$ and $\text{proj}_\theta y$ such that $z \notin \text{proj}_\theta F$. Thus the line L through z and perpendicular to L_θ does not intersect F, so if U and V are the two open half-planes bounded by L, we have $F = (F \cap U) \cup (F \cap V)$ with $x \in F \cap U$ and $y \in F \cap V$. Thus x and y are in different connected components of F, contradicting that F is connected. We conclude that $\mathcal{L}^1(\text{proj}_\theta F) > 0$ for all directions except the direction of the segment $[x, y]$.

In fact, we may conclude that if F is connected and contains more than one point then $\mathcal{L}^1(\text{proj}_\theta F) > 0$, and indeed that $\text{proj}_\theta F$ is an interval of positive length, for all θ unless F is a subset of a straight line, in which case this is true for all but one direction θ.

6.8 Write L_θ for the line through the origin in direction θ. Then $\text{proj}_\theta(x, y)$ is the point on L_θ at distance $x \cos\theta + y \sin\theta = (x + y\lambda)\cos\theta$ from the origin, where $\lambda = \tan\theta$. Thus for all θ such that $\cos\theta \ne 0$, the set $E + \lambda F$ is similar to $\text{proj}_\theta(E \times F)$, so for almost all θ, that is for almost all λ,

$$\dim_{\text{H}}(E + \lambda F) = \dim_{\text{H}} \text{proj}_\theta(E \times F) = \min\{1, \dim_{\text{H}} E \times F\}$$

by Theorem 6.1.

6.9 Let $F = \bigcup_{i=1}^{\infty} F_i$ be a countable union of the irregular 1-sets F_i. By Theorem 6.4, $\mathcal{L}^1(\mathrm{proj}_\theta F_i) = 0$ for almost all θ, say for all $\theta \notin \Theta_i$, where $\mathcal{L}^1(\Theta_i) = 0$. Thus

$$\mathcal{L}^1(\mathrm{proj}_\theta F) = \mathcal{L}^1\left(\mathrm{proj}_\theta \bigcup_{i=1}^{\infty} F_i\right) \leq \sum_{i=1}^{\infty} \mathcal{L}^1(\mathrm{proj}_\theta F_i) = 0$$

for all $\theta \notin \bigcup_{i=1}^{\infty} \Theta_i$, where $\mathcal{L}^1\left(\bigcup_{i=1}^{\infty} \Theta_i\right) \leq \sum_{i=1}^{\infty} \mathcal{L}^1(\Theta_i) = 0$. Thus $\mathcal{L}^1(\mathrm{proj}_\theta F) = 0$ for almost all θ.

6.10 Suppose $\mathcal{H}^1(C \cap (E \times F)) > 0$ for some rectifiable curve C. Then $C \cap (E \times F)$ is a regular 1-set by Proposition 5.6, so $\mathcal{L}^1(\mathrm{proj}_\theta(C \cap (E \times F))) > 0$ for all except at most one direction θ. Thus either $0 < \mathcal{L}^1(\mathrm{proj}_0(C \cap (E \times F))) \leq \mathcal{L}^1(\mathrm{proj}_0(E \times F)) \leq \mathcal{L}^1(E)$ or $0 < \mathcal{L}^1(\mathrm{proj}_{\pi/2}(C \cap (E \times F))) \leq \mathcal{L}^1(\mathrm{proj}_{\pi/2}(E \times F)) = \mathcal{L}^1(F)$, a contradiction.

6.11 This result of this exercise may be obtained by transforming Projection theorem 6.1 under a projective transformation.

In the plane, let C be the 'line at infinity', that is the set of directions of lines in the plane. We claim that, given a line L in $\mathbb{R}^2$, there exists a natural bijection $\psi : \mathbb{R}^2 \cup C \to \mathbb{R}^2 \cup C$ such that $\psi(L) = C$ which has nice geometrical properties regarding straight lines, projections and dimensions.

Regard $\mathbb{R}^2$ as the 'x-y'coordinate plane in $\mathbb{R}^3$, let C be its line at infinity, and let L be a given line in $\mathbb{R}^2$. Let $(a, b, 0)$ be some point of L and let p be the point $(a, b, 1)$. Let P be a plane which is perpendicular to $\mathbb{R}^2$ and parallel to L but not containing L, and let C' be the line at infinity of P. Define $\psi : \mathbb{R}^2 \cup C \to P \cup C'$ by taking $\psi(x)$ to be the point of intersection of P with the line through x and p. If $x \in L$ then we take this to be the point of the 'line at infinity' C' corresponding to the direction of the line through x and p. The map ψ extends to C by mapping a direction in C onto the point of intersection of P with the line through p in that direction. By identifying P with $\mathbb{R}^2$ we get the desired mapping $\psi : \mathbb{R}^2 \cup C \to \mathbb{R}^2 \cup C$.

It is immediate that (i) ψ maps the set of straight lines (including C) bijectively onto the set of straight lines, (ii) $\psi(L) = C$. For our purposes we note the following particular properties which follow easily form the construction: (iii) ψ is bi-Lipschitz on every set B such that B and $\psi(B)$ are bounded subsets of $\mathbb{R}^2$, so in particular F and $\psi(F)$ have equal Hausdorff dimension (provided they both avoid L and C), (iv) for $E \subset L$ we have $\mathcal{L}^1(\psi(E)) > 0$ (thinking of $\psi(E)$ as a set of directions) if and only if $\mathcal{L}^1(E) > 0$, (v) for each $x \in L$, a set of lines $\{L_\theta : \theta \in \Theta\}$ through x has positive angular measure, i.e. $\mathcal{L}^1(\Theta) > 0$, if and only if the set of parallel image lines $\{\psi(L_\theta) : \theta \in \Theta\}$ in direction $\psi(x)$ have displacements of positive $\mathcal{L}^1$-measure.

We may now deduce the required results by transforming Projection theorem 6.1 under ψ and using the above properties. Let L be a straight line and let ψ be as above, so that $\psi(x) \in C$ if $x \in L$. Let F be a Borel subset of $\mathbb{R}^2 \backslash L$ with $\dim_H F > 1$. Then $\dim_H(\psi(F)) > 1$. By Projection Theorem 6.1(b), for almost all directions ψ the lines in direction $\psi(x)$ that intersect $\psi(F)$ have displacements of positive $\mathcal{L}^1$-measure. Transforming back using (iv) and (v) above, this becomes that for almost all $x \in L$, $\mathcal{L}^1\{$ directions of lines through x that intersect $F\} > 0$.

We may show in a similar manner that if $\dim_H F = s \leq 1$ then for almost all $x \in L$, $\dim_H \mathrm{proj}_x F = s$; here we replace (v) above by: (vi) for each $x \in L$, a set of lines $\{L_\theta : \theta \in \Theta\}$ through x has angular dimension s, i.e. $\dim_H(\Theta) = s$, if and only if the set of parallel image lines $\{\psi(L_\theta) : \theta \in \Theta\}$ in direction $\psi(x)$ have displacements of Hausdorff dimension s.

Chapter 7

7.1 We have that $\dim_H[0, 1] = \overline{\dim}_B[0, 1] = 1$, so by Corollary 7.4

$$\dim_H(F \times [0, 1]) = \dim_H F + \dim_H[0, 1] = \dim_H F + 1.$$

7.2 From Example 4.5 or Exercise 2.14, $\dim_H F_\lambda = \overline{\dim}_B F_\lambda = \log 2/\log(2/(1-\lambda))$, so by Formula 7.5 and Corollary 7.4

$$\frac{\log 2}{\log(2/(1-\lambda))} + \frac{\log 2}{\log(2/(1-\mu))} = \overline{\dim}_B F_\lambda + \overline{\dim}_B F_\mu \geq \overline{\dim}_B(F_\lambda \times F_\mu)$$
$$\geq \dim_H(F_\lambda \times F_\mu) = \dim_H F_\lambda + \dim_H F_\mu = \frac{\log 2}{\log(2/(1-\lambda))}$$
$$+ \frac{\log 2}{\log(2/(1-\mu))}.$$

Hence

$$\dim_H(F_\lambda \times F_\mu) = \dim_B(F_\lambda \times F_\mu) = \frac{2\log 2}{\log(2/(1-\lambda))}.$$

7.3 This is a slight variant on Example 7.7. First assume that $F \subset [a, b]$ for some $0 < a < b < \infty$. The mapping $f : [a, b] \times \mathbb{R} \to \mathbb{R}^2$ given by $f(x, y) = (x \cos y, x \sin y)$ is Lipschitz, with $F' = f(F \times [0, 2\pi])$. Thus

$$\dim_H F' = \dim_H f(F \times [0, 2\pi]) \leq \dim_H(F \times [0, 2\pi])$$
$$= \dim_H F + \dim_H[0, 2\pi] = \dim_H F + 1$$

by Corollary 2.4(a) and Example 7.6.

On the other hand, for $0 < a < b < \infty$, the restriction $f : [a, b] \times [0, \pi] \to \mathbb{R}^2$ is a bi-Lipschitz function, with $F' \supset f(F \times [0, \pi])$. Thus

$$\begin{aligned}\dim_{\mathrm{H}} F' &\geq \dim_{\mathrm{H}} f(F \times [0, \pi]) \geq \dim_{\mathrm{H}}(F \times [0, \pi]) \\ &= \dim_{\mathrm{H}} F + \dim_{\mathrm{H}}[0, \pi] = \dim_{\mathrm{H}} F + 1\end{aligned}$$

by Corollary 2.4(b) and Example 7.6. We conclude that $\dim_{\mathrm{H}} F' = \dim_{\mathrm{H}} F + 1$.

Finally, for $F \subset [0, \infty]$, we have, by the above, that $\dim_{\mathrm{H}} F' \cap ((B(0, n) \setminus B^o(0, 1/n)) = \dim_{\mathrm{H}}(F \cap [1/n, n]) + 1$ for each integer n. Noting that $F\backslash\{0\} = \bigcup_{n=2}^{\infty}(F \cap [1/n, n])$ and using countable stability of Hausdorff dimension, see Section 2.2, we get $\dim_{\mathrm{H}}(F' \setminus \{0\}) = \dim_{\mathrm{H}}(F \setminus \{0\}) + 1$, so adding in the origin if necessary gives $\dim_{\mathrm{H}} F' = \dim_{\mathrm{H}} F + 1$.

Note further, that by using the Lipschitz mapping properties of Hausdorff measures (2.9) in a similar way, if $F \subset [a, b]$ for some $0 < a < b < \infty$ then $0 < \mathcal{H}^s(F') < \infty$, where $s = \dim_{\mathrm{H}} F + 1$.

7.4 Let E be any Borel subset of $[0, 1]$ such that $\dim_{\mathrm{H}} E = 1$ and length$(E) = 0$. (For example, we might take $E = \bigcup_{k=1}^{\infty} E_k$ where E_k are Borel sets with $\dim_{\mathrm{H}} E_k \nearrow 1$, see Exercise 4.9.) Setting $F = E \times E \subset \mathbb{R}^2$, we get, using Product Formula 7.2, that $2 \geq \dim_{\mathrm{H}} F = \dim_{\mathrm{H}}(E \times E) \geq 2\dim_{\mathrm{H}} E = 2$, so $\dim_{\mathrm{H}} F = 2$. Moreover, the projection of F onto each of the coordinate axes is just E, so these projections have zero length.

If F_0 is a 1-set with $F_0 \subset F$, then the projection of F_0 onto each coordinate axis is a subset of E, and so has length 0. Thus F_0 is a 1-set with projections of zero length in two directions, so is irregular by Corollary 6.6.

If C is a rectifiable curve, it follows that $C \cap F$ is both irregular and regular, see after Proposition 5.1, so $\mathcal{H}^1(C \cap F) = 0$, that is $C \cap F$ has zero length.

7.5 Let $\{U_i\}$ and $\{V_j\}$ be δ-covers of E and F by $N_\delta(E)$ and $N_\delta(F)$ cubes respectively. Then $\{U_i \times V_j\}_{i,j}$ is a $\delta\sqrt{n}$-cover of $E \times F$. hence

$$N_{\delta\sqrt{n}}(E \times F) \leq N_\delta(E) N_\delta(F),$$

so

$$\frac{\log N_{\delta\sqrt{n}}(E \times F)}{-\log \delta\sqrt{n}} \leq \frac{\log N_\delta(E)}{-\log \delta - \log \sqrt{n}} + \frac{\log N_\delta(F)}{-\log \delta - \log \sqrt{n}}.$$

Taking upper limits we get

$$\begin{aligned}\overline{\dim}_{\mathrm{B}}(E \times F) &\leq \overline{\lim}_{\delta \to 0} \frac{\log N_{\delta\sqrt{n}}(E \times F)}{-\log \delta\sqrt{n}} \\ &\leq \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta - \log \sqrt{n}} + \overline{\lim}_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta - \log \sqrt{n}}\end{aligned}$$

$$\leq \overline{\lim}_{\delta\to 0}\frac{\log N_\delta(E)}{-\log\delta} + \overline{\lim}_{\delta\to 0}\frac{\log N_\delta(F)}{-\log\delta}$$
$$= \overline{\dim}_B E + \overline{\dim}_B F.$$

7.6 Note that $(x, y) \mapsto (x + y)/\sqrt{2}$ is projection onto the line '$y = x$' and $(x, y) \mapsto (x - y)/\sqrt{2}$ is projection onto the line '$y = -x$'. Thus, with F the middle third Cantor set, $E = \{(x, y) \in \mathbb{R}^2 : x + y \in F \text{ and } x - y \in F\}$ is just $F \times F$ scaled down by a factor of $1/\sqrt{2}$ and rotated by $-\pi/4$ about the origin. In particular, E is similar to $F \times F$. By Example 7.6 and Formula 7.5, $\dim_H(F \times F) = \dim_B(F \times F) = 2\log 2/\log 3$, so as the dimensions are preserved under similarity transformations, $\dim_H E = \dim_B E = 2\log 2/\log 3$.

7.7 Recall that $(x, y) \mapsto (x - y)/\sqrt{2}$ is projection onto the line L given by '$y = -x$'. Hence the difference set $D = \{x - y : x, y \in F\}$ is similar to $\mathrm{proj}_L(F \times F)$, so

$$\dim_H D = \dim_H \mathrm{proj}_L(F \times F) \leq \dim_H(F \times F) = 2\dim_H F,$$

by (6.1) and Corollary 7.4. Since D is a subset of a line, $\dim_H D \leq \min\{1, 2 \dim_H F\}$.

7.8 With F the middle third Cantor set, the set $E = \{(x, y) : y - x^2 \in F\}$ is the union of the parabolae $\{y = x^2 + a : a \in F\}$, that is a stack of homothetic (i.e. translates of each other) parabolae, that intersect the y-axis in the points of F.

Locally, E is the product of a line segment and the Cantor set, so we would expect E to have dimension $1 + \log 2/\log 3$. More formally, defining $\phi(x, y) = (x, y + x^2)$, it is easy to see that, for each k, the mapping $\phi : [-k, k] \times F \to E \cap ([-k, k] \times \mathbb{R})$ is a bi-Lipschitz bijection, so

$$\dim_H(E \cap ([-k, k] \times \mathbb{R})) = \dim_H([-k, k] \times F) = 1 + \log 2/\log 3.$$

Since $E = \cup_{k=1}^{\infty} E \cap ([-k, k] \times \mathbb{R})$, we conclude that $\dim_H E = \log 2/\log 3 + 1$.

Technically, the box dimension of E is not defined since E is unbounded. Any non-trivial bounded portion has box dimension $\log 2/\log 3 + 1$.

7.9 Write L_x for the line through $(x, 0)$ parallel to the y-axis, and let $E_s = \{x \in \mathbb{R} : \dim_H(F \cap L_x) \geq s\}$. By Corollary 7.12, $\dim_H F \geq s + \dim_H E_s$ for all $0 \leq s \leq 1$, so $\dim_H F \geq \sup_{0\leq s\leq 1}\{s + \dim_H E_s\}$.

7.10 Writing E_k for the kth stage of the iterative construction of F in the usual way, we note that E_k consists of 12^k rectangles of size $3^{-k} \times 5^{-k}$. Each of these rectangles may be covered by at most $(5/3)^k + 1 \leq 2(5/3)^k$ squares of side 5^{-k} by dividing the rectangles using a series of vertical cuts. Thus E_k may be covered by $12^k \times 2 \times (5/3)^k = 2 \times 20^k$ squares of side 5^{-k} i.e. of diameter $5^{-k}\sqrt{2}$. In the usual way (see Theorem 4.1) this gives that $\dim_H F \leq \log 20/\log 5 = (\log 5 + \log 4)/\log 5 = 1 + \log 4/\log 5$.

For the lower bound, let L_x be the line through $(x, 0)$ parallel to the y-axis. Then, except for x of the form $j3^{-k}$ where j and k are integers, we have that $E_k \cap L_x$ consists of 4^k intervals of length 5^{-k}. A standard application of the mass distribution principle (considering a mass such that each of these intervals has mass 4^{-k}) gives that $\dim_H(F \cap L_x) \geq \log 4/\log 5$. By Corollary 7.12 $\dim_H F \geq 1 + \log 4/\log 5$, so $\dim_H F = 1 + \log 4/\log 5$.

7.11 Writing E_k for the kth stage of the iterative construction of F in the usual way, we note that E_k consists of 8^k rectangles of size $3^{-k} \times 5^{-k}$. For a given positive integer k, let q be the integer such that $5^{-k-1} < 3^{-q} \leq 5^{-k}$. Then dividing the rectangles of E_k horizontally into nearly square rectangles of size $3^{-q} \times 5^{-k}$ and selecting those above the set C_k, the kth stage of the usual middle third Cantor set construction on the x-axis, we get that F may be covered by $2^q 4^k = 3^{q \log 2/\log 3} 4^k \leq 5^{(k+1)\log 2/\log 3} 4^k$ rectangles of size $3^{-q} \times 5^{-k}$, each contained in a square of diameter $5^{-k}\sqrt{2}$. In the usual way (see theorem 4.1) this gives that $\dim_H F \leq ((\log 2/\log 3)\log 5 + \log 4)/\log 5 = \log 2/\log 3 + \log 4/\log 5$.

The lower bound is similar to Exercise 7.10. Let L_x be the line through $(x, 0)$ parallel to the y-axis. For all $x \in C$ where C is the middle third Cantor set, except those x of the form $j3^{-k}$ where j and k are integers, we have that $E_k \cap L_x$ consists of 4^k intervals of length 5^{-k}. The mass distribution principle (considering a mass such that each of these intervals has mass 4^{-k}) gives that $\dim_H(F \cap L_x) \geq \log 4/\log 5$. By Corollary 7.12 $\dim_H F \geq \dim_H C + \log 4/\log 5 = \log 2/\log 3 + \log 4/\log 5$, so $\dim_H F = \log 2/\log 3 + \log 4/\log 5$.

Chapter 8

8.1 Let E and F be line segments of lengths $\mathcal{L}(E)$ and $\mathcal{L}(F)$ making an angle $\theta \neq 0$ with each other. Then E and $F + x$ intersect if and only if x lies in a parallelogram that is a translate of that formed by the vectors along E and F. Thus

$$\int \#(E \cap (F + x))dx = \text{ area of parallelogram } = \mathcal{L}(E)\mathcal{L}(F)|\sin\theta|.$$

Now letting the direction of F vary before translating,

$$\int \#(E \cap \sigma(F))d\sigma = \mathcal{L}(E)\mathcal{L}(F)\int_0^{2\pi} |\sin\theta| d\theta = 4\mathcal{L}(E)\mathcal{L}(F). \qquad (*)$$

Now let $E = \bigcup_{i=1}^{m} E_i$ and $F = \bigcup_{j=1}^{n} F_j$ be polygonal curves with E_i and F_j line segments of lengths $\mathcal{L}(E_i)$ and $\mathcal{L}(F_j)$. Then by $(*)$

$$\int \#(E \cap \sigma(F))d\sigma = \sum_{i=1}^{m}\sum_{j=1}^{n}\int \#(E_i \cap \sigma(F_j))d\sigma \qquad (1)$$

$$= \sum_{i=1}^{m}\sum_{j=1}^{n} 4\mathcal{L}(E_i)\mathcal{L}(F_j) = 4\mathcal{L}(E)\mathcal{L}(F), \qquad (**)$$

where $\mathcal{L}(E)$ and $\mathcal{L}(F)$ are the total lengths of E and F.

We proceed from polygonal curves to rectifiable curves by approximation. Let E and F be rectifiable curves, and let E_n and F_n be sequences of polygonal curves giving closer and closer approximations to E and F, such that each E_n is a refinement of E_{n-1} (i.e. the polygonal curve E_n is obtained from E_{n-1} by adding further vertices) and such that each F_n is a refinement of F_{n-1}, and such that $\lim_{n\to\infty}\mathcal{L}(E_n) = \mathcal{L}(E)$ and $\lim_{n\to\infty}\mathcal{L}(F_n) = \mathcal{L}(F)$. From $(**)$

$$\int \#(E_n \cap \sigma(F_n))d\sigma = 4\mathcal{L}(E_n)\mathcal{L}(F_n).$$

We now take the limit as $n \to \infty$. Provided that $\int \#(E_n \cap \sigma(F_n))d\sigma \to \int \#(E \cap \sigma(F))d\sigma$ we get that

$$\int \#(E \cap \sigma(F))d\sigma = 4\mathcal{L}(E)\mathcal{L}(F). \qquad (*)$$

for rectifiable E and F. For most specific curves E and F, this will follow from the bounded convergence theorem or the monotone convergence theorem. Justification of this step is more involved for general rectifiable curves, see Santalo (1976).

8.2 Let let L be a unit segment in the plane oriented perpendicular to direction θ. Assume that C contains no line segment parallel to L (this can only happen for a set of directions of $\mathcal{L}^1$-measure 0), and let C^+ and C^- be the 'upper' and 'lower' parts of the curve C with respect to the direction of L. Then $(L + x) \cap C^+$ is a single point if x lies in a region congruent to that swept out by translating C^+ unit distance in the direction of L, otherwise

$(L+x)\cap C^+ = \emptyset$. Thus

$$\int \#(C^+ \cap (L+x))dx = \int \#(C^- \cap (L+x))dx$$
$$= \mathcal{L}^1(\mathrm{proj}_\theta C) \times 1$$

so

$$\int \#(C \cap (L+x))dx = 2\mathcal{L}^1(\mathrm{proj}_\theta C).$$

Integrating with respect to θ for $0 \le \theta < 2\pi$ and using Exercise 8.1 gives

$$4\mathcal{L}^1(C) = \int \#(C \cap \sigma(F))d\sigma = \int_0^{2\pi} 2\mathcal{L}^1(\mathrm{proj}_\theta C)d\theta$$

giving the required formula.

8.3 We recall that, with E the product of two middle third Cantor sets, $\dim_H E = 2\log 2/\log 3 = \log 4/\log 3$.

(i) With F a circle $\dim_H F = \dim_B F = 1$. By Theorem 8.1 and Corollary 7.4

$$\begin{aligned}\dim_H(E \cap (F+x)) &\le \max\{0, \dim_H(E \times F) - 2\}\\ &= \max\{0, \dim_H E + \dim_H F - 2\}\\ &= \log 4/\log 3 + 1 - 2 = \log 4/\log 3 - 1\end{aligned}$$

for almost all $x \in \mathbb{R}^2$, and thus

$$\dim_H(E \cap \sigma(F)) \le \log 4/\log 3 - 1$$

for almost all congruence transformations σ.

By Theorem 8.2 (b)

$$\begin{aligned}\dim_H(E \cap \sigma(F)) \ge \dim_H E + \dim_H F - 2 &= \log 4/\log 3 + 1 - 2\\ &= \log 4/\log 3 - 1\end{aligned}$$

for a set of congruence transformations of positive measure.

(ii) With F the von Koch curve, $\dim_H F = \dim_B F = \log 4/\log 3$. By Theorem 8.1 and Corollary 7.4

$$\dim_H(E \cap (F+x)) \le \max\{0, \dim_H(E \times F) - 2\}$$

$$= \max\{0, \dim_H E + \dim_H F - 2\} = 2\log 4/\log 3 - 2$$

for almost all $x \in \mathbb{R}^2$, and thus

$$\dim_H(E \cap \sigma(F)) \le 2\log 4/\log 3 - 2$$

for almost all congruence transformations σ.

By Theorem 8.2 (a)

$$\begin{aligned}\dim_H(E \cap \sigma(F)) &\ge \dim_H E + \dim_H F - 2\\ &= \log 4/\log 3 + \log 4/\log 3 - 2 = 2\log 4/\log 3 - 2\end{aligned}$$

for a set of similarity transformations of positive measure. Note that we cannot apply Theorem 8.2 (c) since $\dim_H E, \dim_H F \le 3/2$.

(iii) With F the product of two middle third Cantor sets, $\dim_H F = \dim_B F = \log 4/\log 3$. By Theorem 8.1 and Corollary 7.4

$$\begin{aligned}\dim_H(E \cap (F + x)) &\le \max\{0, \dim_H(E \times F) - 2\}\\ &= \max\{0, \dim_H E + \dim_H F - 2\} = 2\log 4/\log 3 - 2\end{aligned}$$

for almost all $x \in \mathbb{R}^2$, and thus

$$\dim_H(E \cap \sigma(F)) \le 2\log 4/\log 3 - 2$$

for almost all congruence transformations σ.

By Theorem 8.2 (a)

$$\dim_H(E \cap \sigma(F)) \ge \dim_H E + \dim_H F - 2 = 2\log 4/\log 3 - 2$$

for a set of similarity transformations of positive measure. Again, we cannot apply Theorem 8.2 (c) since $\dim_H E, \dim_H F \le 3/2$.

8.4 As in Theorem 8.1, we prove this when $n = 1$. Let L_c be the line $x = y + c$. If $\dim_H(E \times F) < 1$, the projection of $E \times F$ onto the line $x + y = 0$ has zero length by (6.1), in other words, $(E \times F) \cap L_c = \emptyset$ for almost all $c \in \mathbb{R}$. But $(E \times F) \cap L_c$ is similar to $E \cap (F + c)$ (since $(x, x - c) \in (E \times F) \cap L_c$ if and only if $x \in E \cap (F + c)$). Thus $E \cap (F + c) = \emptyset$ for almost all $c \in \mathbb{R}$.

8.5 Let E be the set of points (x, y) in the unit square A such that both coordinates x and y are rational. Then $\dim_B E = \dim_B \overline{E} = \dim_B A = 2$, by Proposition 3.4. Let F be a line segment. Since E is a countable set (or since its projection in every direction has length 0), $E \cap \sigma(F) = \emptyset$ so that $\dim_B(E \cap \sigma(F)) = 0$ for almost all similarities σ. However, $\dim_B E + \dim_B F - 2 = 2 + 1 - 2 = 1$, so (8.5) fails for almost all similarities σ.

8.6 Let C be a middle λ Cantor set with λ chosen so that $\dim_{\mathrm{H}} C = s - 1$ (see after Example 4.5) and transformed by a similarity so that the end points are $\frac{1}{2}$ and 1. Let F be the 'target' given in polar coordinates by $F = \{(r, \theta) : r \in C, 0 \leq \theta < 2\pi\}$. By Exercise 7.3, $\dim_{\mathrm{H}} F = 1 + \dim_{\mathrm{H}} C = s$, and by the note at the end of the solution of Exercise 7.3, $0 < \mathcal{H}^s(F) < \infty$, so F is an s-set.

It is easy to see that any line segment E that intersects the interior of the unit disc cuts a set of the rings of the target F corresponding to a similar subset of C, that is $E \cap F$ contains a subset that is bi-Lipschitz equivalent to C and so has positive s-dimensional Hausdorff measure. On the other hand, provided that E is not tangential to one of the rings of F, $E \cap F$ is contained in a finite union of Lipschitz images of C, and so $E \cap F$ has finite s-dimensional Hausdorff measure and so is an s-set.

8.7 Writing L_k for the line segment $\{(x, k^{-1/2}) : 0 \leq x \leq k^{-1/2}\}$, we see that $E = \{(0,0)\} \bigcup_{k=1}^{\infty} L_k$. Given small enough δ, let k be the integer such that

$$\tfrac{1}{2}(k+1)^{-3/2} \leq k^{-1/2} - (k+1)^{-1/2} \leq \delta < (k-1)^{-1/2} - k^{-1/2} \qquad (*)$$

where the left hand inequality follows using the mean value theorem. Then any set of diameter δ or less can intersect at most one of the segments $L_1, \ldots, L_k$, and the segment L_j requires at least $j^{-1/2}/\delta$ sets of diameter δ in any covering. Thus, using an 'integral test' estimate for the sum,

$$N_\delta(E) \geq \sum_{j=1}^{k} \frac{j^{-1/2}}{\delta} \geq \frac{1}{\delta} \int_0^k x^{-1/2} dx = \frac{2k^{1/2}}{\delta} \geq 2\delta^{-1} c\delta^{-1/3} \geq 2c\delta^{-4/3},$$

using $(*)$, where c is independent of δ. It follows immediately that

$$\overline{\dim}_{\mathrm{B}} E \geq \underline{\dim}_{\mathrm{B}} E = \underline{\lim}_{\delta \to 0} \frac{\log N_\delta(E)}{-\log \delta} \geq \frac{\log 2c\delta^{-4/3}}{-\log \delta} = \frac{4}{3}.$$

Every line L, that does not pass through $(0,0)$ and that does not contain one of the line segments L_k, intersects E in a finite set of points, so in particular $\dim_{\mathrm{B}}(L \cap E) = 0$.

8.8 Let $0 < s < 1$, let I be an interval and let $\epsilon > 0$ be given. We use a mass distribution method to estimate $\mathcal{H}^s_\infty(I \cap E_k)$ for large k. We may find $\eta > 0$ such that $\eta^{1-s} \leq \left(\frac{2}{3} + \epsilon\right)$. For given k, let μ be the mass distribution on E_k given by the restriction of Lebesgue measure to E_k. Note that if U is a 'not too small' subinterval of I and k is large enough then $\mu(U)$ is close to $\frac{2}{3}|U|$, since two-thirds of the ternary intervals of length 3^{-k} are present in E_k. In particular, provided that k is large enough, $k \geq k_0$, say, we can ensure that $\left(\frac{2}{3} - \epsilon\right)|I| \leq \mu(I)$ and

$$\mu(U) \leq \left(\frac{2}{3} + \epsilon\right)|U| = \left(\frac{2}{3} + \epsilon\right)\frac{|U|}{|I|}|I| \leq \left(\frac{2}{3} + \epsilon\right)\left(\frac{|U|}{|I|}\right)^s |I|$$
$$\text{if } |U| \geq \eta|I|,$$

and

$$\mu(U) \leq |U| = |U|^s|U|^{1-s} \leq \frac{|U|^s}{|I|^s}\eta^{1-s}|I| \leq \left(\frac{2}{3} + \epsilon\right)\left(\frac{|U|}{|I|}\right)^s |I|$$
$$\text{if } |U| \leq \eta|I|.$$

Thus if $\{U_i\}$ is any cover of $I \cap E_k$, then

$$\left(\frac{2}{3} - \epsilon\right)|I| \leq \mu(I) \leq \sum_i \mu(U_i) \leq \left(\frac{2}{3} + \epsilon\right)|I|\frac{\sum_i |U_i|^s}{|I|^s}$$

giving that

$$\sum_i |U_i|^s \geq |I|^s \frac{\left(\frac{2}{3} - \epsilon\right)}{\left(\frac{2}{3} + \epsilon\right)} \geq |I|^s(1 - 2\epsilon).$$

Thus $\mathcal{H}^s_\infty(I \cap E_k) \geq |I|^s(1 - 2\epsilon)$.

It follows by (8.8) that $\lim_{k\to\infty} \mathcal{H}^s_\infty(I \cap E_k) = |I|^s$, so $F = \overline{\lim}_{k\to\infty} E_k$ is in class $\mathcal{C}^s(-\infty, \infty)$ for all $0 < s < 1$.

It is immediate from Proposition 8.5 that $\dim_{\mathrm{H}} F \geq s$ for all $0 < s < 1$, so $\dim_{\mathrm{H}} F = 1$. Moreover, for all $0 < s < 1$ and any $x_1, x_2, \ldots$, we have that $F + x_i \in \mathcal{C}^s(-\infty, \infty)$, by Proposition 8.8 or by repeating the argument above. Thus, by Proposition 8.6 and Corollary 8.7, $\cap_{i=1}^\infty (F + x_i) \in \mathcal{C}^s(-\infty, \infty)$ and $\dim_{\mathrm{H}} \cap_{i=1}^\infty (F + x_i) \geq s$ for all $0 < s < 1$, so we conclude that $\dim_{\mathrm{H}} \cap_{i=1}^\infty (F + x_i) = 1$.

Chapter 9

9.1 The Hausdorff metric is given by

$$d(A, B) = \inf\{\delta : A \subset B_\delta \text{ and } B \subset A_\delta\},$$

where A_δ, B_δ are the δ-neighbourhoods of A and B.

To show that d satisfies condition (i) for a metric, we first note that $d(A, B) \geq 0$ with $d(A, A) = 0$. Now suppose that $d(A, B) = 0$. Taking $x \in B$, we note that, for each positive integer n, we must have $x \in B \subset A_{1/n}$ and so there exists $x_n \in A$ such that $|x - x_n| \leq 1/n$. Thus $x \in \overline{A} = A$ since A is compact and hence closed. Thus $B \subset A$; similarly $A \subset B$ so $A = B$ as required.

Clearly by the symmetry of the definition, $d(A, B) = d(B, A)$, which is condition (ii) for a metric.

To show that d satisfies condition (iii), we suppose that $d(A, C) = \epsilon_1$ and $d(C, B) = \epsilon_2$. Then, for each $\delta_1 > \epsilon_1$ and each $\delta_2 > \epsilon_2$, we have

$$A \subset C_{\delta_1},\ C \subset A_{\delta_1},\ C \subset B_{\delta_2},\ B \subset C_{\delta_2}.$$

So

$$A \subset C_{\delta_1} \subset B_{\delta_1+\delta_2} \text{ and } B \subset C_{\delta_2} \subset A_{\delta_2+\delta_1}.$$

Thus $d(A, B) \leq \delta_1 + \delta_2$ for all δ_1, δ_2 with $\delta_1 > \epsilon_1$ and $\delta_2 > \epsilon_2$, so $d(A, B) \leq \epsilon_1 + \epsilon_2 = d(A, C) + d(C, B)$ as claimed.

9.2 Let c be any real number for which $0 < c < 1$. Then the interval $[0, 1]$ is the attractor for the similarity transformations defined on $\mathbb{R}$ by $S_1(x) = cx$ and $S_2(x) = (1 - c)x + c$, since $S_1([0, 1]) \cup S_2([0, 1]) = [0, c] \cup [c, 1] = [0, 1]$. Clearly S_1 and S_2 are both contractions, so from Theorem 9.1 $[0, 1]$ is the unique non-empty compact attractor for S_1 and S_2.

9.3 We begin by noting that the middle third Cantor set is non-empty and compact. The middle third Cantor set is therefore the attractor for the following four similarity transformations on $\mathbb{R}$ which map the interval $E_0 = [0, 1]$ onto the four intervals in E_2:

$$S_1(x) = x/9,\ S_2(x) = x/9 + 2/9,\ S_3(x) = x/9 + 2/3,\ S_4(x) = x/9 + 8/9.$$

The ratios of these similarities are all $1/9$ so that equation (9.13) is $4(1/9)^s = 1$. Taking the log of both sides gives $\log 4 - s \log 9 = 0$ so that

$$s = \frac{\log 4}{\log 9} = \frac{\log 2^2}{\log 3^2} = \frac{2\log 2}{2\log 3} = \frac{\log 2}{\log 3}$$

The middle third Cantor set is also the attractor for the following three similarity transformations on $\mathbb{R}$ which map the interval $E_0 = [0, 1]$ onto the first two intervals in E_2 and the second interval in E_1:

$$S_1(x) = x/9,\ S_2(x) = x/9 + 2/9,\ S_3(x) = x/3 + 2/3.$$

The ratios of the similarities are $1/9$, $1/9$ and $1/3$ respectively, so that equation (9.13) is $2(1/9)^s + (1/3)^s = 1$. Putting $x = (1/3)^s$, we can write this as $2x^2 + x = 1$ or, equivalently, $(2x-1)(x+1) = 0$ which has solutions $x = 1/2$ and $x = -1$. Since $x = (1/3)^s > 0$, it follows that $(1/3)^s = 1/2$. Taking logarithms of both sides gives $-s\log 3 = -\log 2$ so that $s = \log 2/\log 3$.

9.4 Recall that the matrix which represents a rotation about the origin through angle θ is $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$. Note that the generator of the von Koch curve (see figure 0.2) has vertices $(0,0)$, $\left(\frac{1}{3}, 0\right)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$, $\left(\frac{2}{3}, 0\right)$ and $(1,0)$. Regarding the similarities that maps the line segment joining $(0,0)$ and $(1,0)$ onto the intermediate segments as a composition of a rotation by $\pm\pi/3$ (if necessary), a scaling by factor of $\frac{1}{3}$ and a translation, we see that an IFS $\{S_1, S_2, S_3, S_4\}$ that has the von Koch curve as attractor is given by

$$S_1\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix}$$

$$S_2\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{3} \\ 0 \end{pmatrix}$$

$$S_3\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{6} \end{pmatrix}$$

$$S_4\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{3}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \frac{2}{3} \\ 0 \end{pmatrix}.$$

The open set condition holds, taking the open set V to be the interior of the isosceles triangle with vertices $(0,0)$, $\left(\frac{1}{2}, \frac{\sqrt{3}}{6}\right)$ and $(1,0)$. This open triangle is mapped by $S_1, \ldots, S_4$ to four similar open triangles at scale $\frac{1}{3}$ with bases on the four segments of the generator, with their union contained in V. Theorem 9.3 immediately gives that the box and Hausdorff dimension s of the von Koch curve is given by $\sum_{i=1}^{4}(1/3)^s = 1$, that is $4 \times 3^{-s} = 1$ or $s = \log 4/\log 3$.

9.5 The set F in figure 0.5 is the attractor of the following five similarities on $\mathbb{R}^2$ which map E_0 onto the five squares in E_1:

$$S_1(x,y) = (x/4, y/4), \quad S_2(x,y) = (x/4 + 3/4, y/4),$$

$$S_3(x,y) = (x/4 + 3/4, y/4 + 3/4), \quad S_4(x,y) = (x/4, y/4 + 3/4)$$

$$S_5(x,y) = (x/2 + 1/4, y/2 + 1/4).$$

The ratios of these similarities are $1/4, 1/4, 1/4, 1/4, 1/2$ respectively. Each of the sets in E_k is compact. Thus F is the intersection of a decreasing sequence of compact sets and is hence compact. So F is the attractor satisfying $F = \bigcup_{i=1}^{5} S_i(F)$. The open set condition holds, taking V as the interior of the initial square E_0, and so it follows from Theorem 9.3 that $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = s$, where s is given by $1 = 4(1/4)^s + (1/2)^s$.

Note that this can be solved by putting $x = (1/2)^s$ to give

$$4x^2 + x - 1 = 0$$

so that

$$x = (-1 \pm \sqrt{17})/8.$$

Since $x = (1/2)^s > 0$, it follows that $(1/2)^s = -1/8 + \sqrt{17}/8$ and so $-s \log 2 = \log(-1/8 + \sqrt{17}/8)$. Thus

$$s = \frac{-\log(-1/8 + \sqrt{17}/8)}{\log 2} = 1.357\ldots.$$

9.6 The set F is the attractor for the three similarities on $\mathbb{R}^2$:

$$S_1(x, y) = (x/2, y/2), \quad S_2(x, y) = (x/2 + 1/2, y/2),$$
$$S_3(x, y) = (y/4 + 1/2, -x/4).$$

These have ratios $1/2, 1/2, 1/4$ respectively. The open set condition holds, taking V to be the interior of the triangle formed by the three free ends of the segments. From Theorem 9.3 $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = s$, where

$$1 = 2(1/2)^s + (1/4)^s = 2(1/2)^s + (1/2)^{2s}.$$

Putting $x = (1/2)^s$, we have

$$x^2 + 2x - 1 = 0$$

and so

$$x = -1 \pm \sqrt{2}.$$

Since $x = (1/2)^s > 0$, it follows that $(1/2)^s = -1 + \sqrt{2}$ and so $-s \log 2 = \log(-1 + \sqrt{2})$. Thus

$$s = \frac{-\log(-1 + \sqrt{2})}{\log 2} = 1.271\ldots.$$

9.7 The set F is the attractor for the following similarities on $\mathbb{R}$ which map $[0, 1]$ onto the intervals $[0, 1/4]$ and $[1/2, 1]$ respectively:

$$S_1(x) = x/4 \text{ and } S_2(x) = x/2 + 1/2,$$

with ratios $1/4$ and $1/2$. The set F is the intersection of a decreasing sequence of non-empty compact sets and is hence non-empty and compact. The open set condition holds, taking V as the open interval $(0, 1)$ and so, by Theorem 9.3, $\dim_{\mathrm{H}} F = \dim_{\mathrm{B}} F = s$, where s is given by

$$1 = (1/4)^s + (1/2)^s = (1/2)^{2s} + (1/2)^s.$$

Putting $x = (1/2)^s$, we have $x^2 + x - 1 = 0$ and so $x = (-1 \pm \sqrt{5})/2$. Since $x = (1/2)^s > 0$, it follows that $(1/2)^s = (-1 + \sqrt{5})/2$.

and so

$$s = \frac{-\log\left(\dfrac{-1+\sqrt{5}}{2}\right)}{\log 2} = 1 - \frac{\log(-1+\sqrt{5})}{\log 2} = 0.6942\ldots.$$

9.8 In each case it may be verified trivially that the stated attractor is compact and satisfies $F = S_1(F) \cup S_2(F)$.

(i) The middle half Cantor set (i.e. the Cantor-type construction with the (open) middle half of intervals removed at each stage).

(ii) The interval $[0, 1]$.

(iii) The interval $[0, 1]$. (Notice that in this case the two parts $S_1([0, 1])$ and $S_2([0, 1])$ overlap non-trivially).

9.9 The open set condition holds for the IFS $\{S_1, \ldots, S_m\}$, taking V as the open unit square, so that the $S_i(V)$ are the interiors of the squares selected in E_1, with $V \supset \bigcup_{i=1}^m S_i(V)$ and the union disjoint.

Thus by Theorem 9.3, the box and Hausdorff dimension s of F is given by $\sum_{i=1}^m (1/p)^s = 1$, that is $m \times p^{-s} = 1$ or $s = \log m/\log p$.

9.10 This is similar to Example 9.8. Here S_1 and S_2 are contractions on the closed set $D = [0, 1] \subset \mathbb{R}$. We note that

$$S_1'(x) = \frac{2}{(2+x)^2} > 0 \text{ and } S_2'(x) = \frac{-2}{(2+x)^2} < 0.$$

Thus S_1 is increasing on D and S_2 is decreasing on D so that

$$S_1(D) = [S_1(0), S_1(1)] = [0, 1/3] \text{ and } S_2(D) = [S_2(1), S_2(0)] = [2/3, 1].$$

Since $F \subset D$, it follows that $S_1(F) \subset [0, 1/3]$ and $S_2(F) \subset [2/3, 1]$. Thus F is the disjoint union of $S_1(F)$ and $S_2(F)$ and so we can apply Propositions 9.6 and 9.7 to estimate $\dim_H F$. (We could use Proposition 9.6 even if the union was not disjoint.)

For $x \in D$, $i = 1, 2$, we have

$$\frac{2}{9} = \frac{2}{3^2} \leq |S_i'(x)| = \frac{2}{(2+x)^2} \leq \frac{2}{2^2} = \frac{1}{2}.$$

It follows from the mean-value theorem that, for $x, y \in D$, $i = 1, 2$,

$$\frac{2}{9}|x-y| \leq |S_i(x) - S_i(y)| \leq \frac{1}{2}|x-y|.$$

By Propositions 9.6 and 9.7, $t \leq \dim_H F \leq s$, where

$$2(2/9)^t = 1 = 2(1/2)^s.$$

Clearly $s = 1$ and taking logs gives

$$t = \frac{\log 1/2}{\log 2/9} = 0.46$$

to two decimal places, so $0.46 < \dim_H F \leq 1$.

These estimates are rather poor, and so we use the fact that F is also the attractor of the four contractions defined by

$$S_1 \circ S_1(x) = \frac{\dfrac{x}{2+x}}{2 + \dfrac{x}{2+x}} = \frac{x}{4+3x} \qquad S_1 \circ S_2(x) = \frac{\dfrac{2}{2+x}}{2 + \dfrac{2}{2+x}} = \frac{1}{3+x}$$

$$S_2 \circ S_1(x) = \frac{2}{2 + \dfrac{x}{2+x}} = \frac{4+2x}{4+3x} \qquad S_2 \circ S_2(x) = \frac{2}{2 + \dfrac{2}{2+x}} = \frac{2+x}{3+x}.$$

Thus for $x \in D$

$$\frac{4}{49} \leq |(S_1 \circ S_1)'(x)| = \left|\frac{4}{(4+3x)^2}\right| \leq \frac{1}{4}$$

$$\frac{1}{16} \leq |(S_1 \circ S_2)'(x)| = \left|\frac{-1}{(3+x)^2}\right| \leq \frac{1}{9}$$

$$\frac{4}{49} \leq |(S_2 \circ S_1)'(x)| = \left|\frac{-4}{(4+3x)^2}\right| \leq \frac{1}{4}$$

$$\frac{1}{16} \leq |(S_2 \circ S_2)'(x)| = \left|\frac{1}{(3+x)^2}\right| \leq \frac{1}{9}.$$

By the mean-value theorem, for each $x, y \in D$, $i = 1, 2$,

$$\frac{4}{49}|x-y| \le |S_i \circ S_1(x) - S_i \circ S_1(y)| \le \frac{1}{4}|x-y|$$

and

$$\frac{1}{16}|x-y| \le |S_i \circ S_2(x) - S_i \circ S_2(y)| \le \frac{1}{9}|x-y|.$$

Since $S_1(F)$ and $S_2(F)$ are disjoint, the sets $S_1 \circ S_1(F)$, $S_1 \circ S_2(F)$, $S_2 \circ S_1(F)$ and $S_2 \circ S_2(F)$ are also disjoint and so it follows from Propositions 9.6 and 9.7 that $t \le \dim_H F \le s$, where

$$2(4/49)^t + 2(1/16)^t = 1 = 2(1/4)^s + 2(1/9)^s.$$

To two decimal places, this is satisfied by $s = 0.80$ and $t = 0.53$ and so $0.52 < \dim_H F < 0.81$.

9.11 We use the notation of Theorem 9.3. If $x \in F$ then, as in (9.7), we have that $x = \bigcap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(E)$. Given $0 < r < |F|$, with $\mathcal{Q}_1$ as in the proof of Theorem 9.3, we have

$$F \cap B(x,r) \subset \bigcup_{i_1,\ldots,i_k \in \mathcal{Q}_1} \overline{V}_{i_1,\ldots,i_k},$$

so

$$\begin{aligned}\mathcal{H}^s(F \cap B(x,r)) &\le \sum_{\mathcal{Q}_1} \mathcal{H}^s(F \cap \overline{V}_{i_1,\ldots,i_k}) = \sum_{\mathcal{Q}_1} \mathcal{H}^s(F_{i_1,\ldots,i_k}) \\ &\le \sum_{\mathcal{Q}_1} (c_{i_1} \cdots c_{i_k})^s |F|^s \le q r^s |F|^s\end{aligned}$$

so

$$\overline{D}^s(F,x) = \overline{\lim}_{r\to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s} \le \overline{\lim}_{r\to 0} \frac{q r^s |F|^s}{(2r)^s} \le q 2^{-s}|F|^s.$$

On the other hand, if $x = \bigcap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(E)$, then choosing k such that $(\min_i c_i) r \le c_{i_1} \cdots c_{i_k}|F| \le r$, we have $F_{i_1,\ldots,i_k} \subset B(x,r)$, so that

$$\begin{aligned}\mathcal{H}^s(F \cap B(x,r)) &\ge \mathcal{H}^s(F_{i_1,\ldots,i_k}) \ge (c_{i_1} \cdots c_{i_k})^s \mathcal{H}^s(F) \\ &\ge (\min_i c_i)^s |F|^{-s} r^s \equiv b r^s\end{aligned}$$

Thus

$$\underline{D}^s(F,x) = \underline{\lim}_{r\to 0} \frac{\mathcal{H}^s(F \cap B(x,r))}{(2r)^s} \ge \underline{\lim}_{r\to 0} \frac{b r^s}{(2r)^s} \ge b 2^{-s}.$$

9.12 Since $F \cap V \subset F$ we have $\dim_{\mathrm{H}}(F \cap V) \leq \dim_{\mathrm{H}} F$.

Since V is an open set intersecting F, there is a bi-Lipschitz mapping $S : F \to V$, so that $F \cap V$ contains a bi-Lipschitz image of F and thus $\dim_{\mathrm{H}}(F \cap V) \geq \dim_{\mathrm{H}} F$. (To see this, let $x \in F \cap V$ so that $x = \bigcap_{k=1}^{\infty} S_{i_1} \circ \cdots \circ S_{i_k}(E)$, as in (9.7). Then $S_{i_1} \circ \cdots \circ S_{i_k}(F) \subset F \cap V$ if k is large enough, so we may take $S = S_{i_1} \circ \cdots \circ S_{i_k}$ as the bi-Lipschitz mapping.) Thus $\dim_{\mathrm{H}}(F \cap V) = \dim_{\mathrm{H}} F$. An identical argument shows that $\underline{\dim}_{\mathrm{B}}(F \cap V) = \underline{\dim}_{\mathrm{B}} F$ and $\overline{\dim}_{\mathrm{B}}(F \cap V) = \overline{\dim}_{\mathrm{B}} F$ since these dimensions are also preserved under bi-Lipschitz mappings.

By Corollary 3.9, $\dim_{\mathrm{P}} F = \overline{\dim}_{\mathrm{B}} F$.

9.13 Note that this is a generalization of Example 7.13 and Exercises 7.10 and 7.11.

(a) The formula of Example 9.11 gives

$$\dim_{\mathrm{H}} F = \log\left(\sum_{j=1}^{p} N_j^{\log p/\log q}\right)\frac{1}{\log p} = \frac{\log(pN^{\log p/\log q})}{\log p} = 1 + \frac{\log N}{\log q}.$$

To check this, write E_k for the kth stage of the iterative construction of F in the usual way, and note that E_k consists of $(pN)^k$ rectangles of size $p^{-k} \times q^{-k}$. Each of these rectangles may be covered by at most $(q/p)^k + 1 \leq 2(q/p)^k$ squares of side q^{-k} by dividing the rectangles using a series of vertical cuts. Thus E_k may be covered by $(pN)^k 2(q/p)^k = 2(Nq)^k$ squares of side q^{-k} i.e. of diameter $q^{-k}\sqrt{2}$. In the usual way (see Theorem 4.1) this gives that $\dim_{\mathrm{H}} F \leq \log(Nq)/\log q = (\log N + \log q)/\log q = 1 + \log N/\log q$.

For the lower bound, let L_x be the line through $(x, 0)$ parallel to the y-axis. Then, except for x of the form jp^{-k} where j and k are integers, we have that $E_k \cap L_x$ consists of N^k intervals of length q^{-k}. A standard application of the mass distribution principle (considering a mass such that each of these intervals has mass N^{-k}) gives that $\dim_{\mathrm{H}}(F \cap L_x) \geq \log N/\log q$. By Corollary 7.12 $\dim_{\mathrm{H}} F \geq 1 + \log N/\log q$, so $\dim_{\mathrm{H}} F = 1 + \log N/\log q$.

(b) The formula of Example 9.11 gives

$$\begin{aligned}\dim_{\mathrm{H}} F &= \log\left(\sum_{j=1}^{p} N_j^{\log p/\log q}\right)\frac{1}{\log p}\\ &= \frac{\log(p_1 N^{\log p/\log q})}{\log p} = \frac{\log p_1}{\log p} + \frac{\log N}{\log q}.\end{aligned}$$

To check this, write E_k for the kth stage of the iterative construction of F in the usual way. Note that E_k consists of $(p_1 N)^k$ rectangles of size $p^{-k} \times q^{-k}$. Let C be the projection of F onto the x-axis, so that C is a self-similar set subset of the x-axis formed by p_1 similarities of ratios $1/p$. Let C_k be the projection of E_k onto the x-axis, so that C_k is the kth stage of the construction of C under the usual process. For a given positive integer k, let s be the integer such that $q^{-k-1} < p^{-s} \le q^{-k}$. Considering the part of F above the set C_s, we get that F may be covered by $p_1^s N^k = p_1^{s \log p_1/\log p} N^k \le q^{(k+1)\log p_1/\log p} N^k$ rectangles of size $p^{-s} \times q^{-k}$, each contained in a square of diameter $q^{-k}\sqrt{2}$. In the usual way (see Theorem 4.1) this gives that $\dim_{\mathrm{H}} F \le \log(q^{\log p_1/\log p} N)/\log q = \log N/\log q + \log p_1/\log p$.

The lower bound is similar to part (a). Let L_x be the line through $(x, 0)$ parallel to the y-axis. Let C be the projection of F onto the x-axis, as above, so that C is a self-similar set subset of the x-axis formed by p_1 similarities of ratios $1/p$.

For all $x \in C$, except those x of the form jp^{-k} where j and k are integers, we have that $E_k \cap L_x$ consists of N^k intervals of length q^{-k}. The mass distribution principle (considering a mass such that each of these intervals has mass N^{-k}) gives that $\dim_{\mathrm{H}}(F \cap L_x) \ge \log N/\log q$. By Corollary 7.12 $\dim_{\mathrm{H}} F \ge \dim_{\mathrm{H}} C + \log N/\log q = \log p_1/\log p + \log N/\log q$, so $\dim_{\mathrm{H}} F = \log p_1/\log p + \log N/\log q$.

9.14 We apply the formulae in Example 9.11 with:

$$p = 3, q = 6, N_1 = 4, N_2 = 1, N_3 = 3, p_1 = 3.$$

Thus, writing $\alpha = \log p/\log q = \log 3/\log 6$,

$$\dim_{\mathrm{H}} F = \log\left(\sum_{j=1}^{p} N_j^{\log p/\log q}\right)\frac{1}{\log p} = \frac{\log(4^\alpha + 1^\alpha + 3^\alpha)}{\log 3} = 1.518\ldots$$

and

$$\dim_{\mathrm{B}} F = \frac{\log p_1}{\log p} + \log\left(\frac{1}{p_1}\sum_{j=1}^{p} N_j\right)\frac{1}{\log q}$$
$$= 1 + \frac{\log\left(\frac{1}{3}(4+1+3)\right)}{\log 5} = 1.627\ldots.$$

Chapter 10

10.1 F is the (non-empty compact) attractor of the IFS $\{S_1, \ldots, S_5\}$ where $S_i = \frac{1}{10}x + \frac{i-1}{5} \quad (i = 1, \ldots, 5)$.

Since $\bigcup_{i=1}^{5} S_i(0,1) = \bigcup_{i=1}^{5} S_i(\frac{i-1}{5}, \frac{i-1}{5} + \frac{1}{10}) \subset (0,1)$, the open set condition holds with open set $(0,1)$, so by Theorem 9.3 $\dim_{\mathrm{H}} F = s$, where $5 \times (1/10)^s = 1$, that is $\dim_{\mathrm{H}} F = \log 5/\log 10$.

10.2 Let $\{S_1, \ldots, S_m\}$ be given by $S_i = \frac{1}{m}x + \frac{i-1}{m}$ $(i = 1, \ldots, m)$. Then, since membership of $F(p_0, \ldots, p_{m-1})$ is determined by the base-m digits of a number after any given place,

$$\begin{aligned}\bigcup_{i=1}^{m} S_i(F(p_0, \ldots, p_{m-1})) &= \bigcup_{i=1}^{m} [0,1) \cap F(p_0, \ldots, p_{m-1}) \\ &= F(p_0, \ldots, p_{m-1}),\end{aligned}$$

that is $F(p_0, \ldots, p_{m-1})$ is a (non-compact) attractor of the S_i.

10.3 With the notation of Section 10.1, the numbers in $\dim_{\mathrm{H}} F(1-3p, p, 2p)$ have twice as many 2s as 1s for all $0 < p < \frac{1}{3}$. Thus we must find the maximum value of $\dim_{\mathrm{H}} F(1-3p, p, 2p)$ over such p. Proposition 10.1 gives

$$\begin{aligned}\phi(p) &\equiv \dim_{\mathrm{H}} F(p, 2p, 1-3p) \\ &= -\frac{1}{\log 3}[(1-3p)\log(1-3p) + p\log p + 2p\log 2p] \\ &= -\frac{1}{\log 3}[p(\log p + 2\log 2p - 3\log(1-3p)) + \log(1-3p)].\end{aligned}$$

Then

$$\frac{d\phi}{dp} = -\frac{1}{\log 3}[\log p + 2\log 2p - 3\log(1-3p)] = -\frac{1}{\log 3}\log\frac{4p^3}{(1-3p)^3}.$$

Thus a maximum occurs when $4p^3 = (1-3p)^3$ or $p = 1/(3+4^{1/3})$, that is when$(1-3p) = 4^{1/3}/(3+4^{1/3})$. The value of the maximum is

$$\phi(1/(3+4^{1/3})) = \frac{1}{\log 3}\left[\log(3+4^{1/3}) - \frac{2}{3}\log 2\right] = 0.9660\ldots.$$

Thus the required Hausdorff dimension is $0.9660\ldots$.

10.4 (i) To find the continued fraction expansion of $41/9$, we first note that

$$\frac{41}{9} = 4 + \frac{5}{9} = 4 + \frac{1}{9/5}$$

so $a_0 = 4$ and $x_1 = 9/5$. Now

$$\frac{9}{5} = 1 + \frac{4}{5} = 1 + \frac{1}{5/4}$$

so $a_1 = 1$ and $x_2 = 5/4$. Now

$$\frac{5}{4} = 1 + \frac{1}{4}$$

so $a_2 = 1$ and $a_3 = 4$. So

$$\frac{41}{9} = 4 + \frac{1}{1+}\frac{1}{1+}\frac{1}{4}.$$

(ii) To find the continued fraction expansion of $\sqrt{5}$, we first note that $2 < \sqrt{5} < 3$ and so

$$\sqrt{5} = 2 + \frac{1}{x_1},$$

where $x_1 > 1$ (that is, $a_0 = 2$). Now

$$x_1 = \frac{1}{\sqrt{5}-2} = \frac{1}{\sqrt{5}-2}\frac{\sqrt{5}+2}{\sqrt{5}+2} = \sqrt{5}+2$$

and so

$$x_1 = 4 + \frac{1}{x_2},$$

where $x_2 > 1$ (that is, $a_1 = 4$). Now

$$x_2 = \frac{1}{x_1 - 4} = \frac{1}{\sqrt{5}-2} = x_1$$

and so

$$x_2 = 4 + \frac{1}{x_3},$$

where $x_3 = x_2 > 1$ (that is, $a_2 = 4$). This process now repeats itself giving $4 = a_3 = a_4 = \cdots$. Thus

$$\sqrt{5} = 1 + \frac{1}{4+}\frac{1}{4+}\frac{1}{4+\cdots}.$$

10.5 Letting $x = 1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+\cdots}$ we see that $x = 1 + 1/(1 + \frac{1}{1+}\frac{1}{1+\cdots}) = 1 + 1/x$. Thus $x^2 - x - 1 = 0$, so $x = \frac{1+\sqrt{5}}{2}$, the golden mean. (We take the positive root of the quadratic equation since x is clearly positive.)

10.6 We have $\sqrt{2} = 1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+\cdots}$ so that curtailing after each term gives successive approximations to $\sqrt{2} = 1.41421\ldots$ of:

$$\frac{3}{2} = 1.5, \quad \frac{7}{5} = 1.4, \quad \frac{17}{12} = 1.41666\ldots,$$
$$\frac{41}{29} = 1.41379\ldots, \quad \frac{99}{70} = 1.41428\ldots$$

(Compare this with $\sqrt{2} = 1.4142136$ to 7 decimal places.)

10.7 This similar to Example 10.2, using Example 9.8. Let F denote the set of positive numbers with infinite continued fraction expansions which have all partial quotients equal to 2 or 3. Then each $x \in F$ can be written as

$$x = a_0 + \frac{1}{x_1},$$

where a_0 is equal to 2 or 3 and $x_1 > 1$, so $2 < x < 4$. Now let $S_1, S_2 : [2,4] \to [2,4]$ be given by

$$S_1(x) = 2 + \frac{1}{x} \text{ and } S_2(x) = 3 + \frac{1}{x}.$$

We claim that F is the attractor of S_1 and S_2. To see this, we note that from the definition of F and the continued fractions, we have that $x \in F$ if and only if either $S_1(x) \in F$ or $S_2(x) \in F$. Thus $F = S_1(F) \cup S_2(F)$. Clear $F \subset [2,4]$ is bounded, and non-empty, since $1 + \frac{1}{2+}\frac{1}{2+}\frac{1}{2+\cdots} \in F$. To see that F is closed, note that its complement is open, since if

$$x = a_0 + 1/(a_1 + 1/(a_2 + 1/(a_3 + \cdots))) \notin F$$

then $a_k \neq 2, 3$ for some k, so numbers whose continued fraction expansion start

$$a_0 + 1/(a_1 + 1/(a_2 + 1/(\cdots + 1/(a_k)))),$$

that is numbers close enough to x, are not in F.

Noting that $S_1([2,4]) = \left[2\frac{1}{4}, 2\frac{1}{2}\right]$ and $S_2([2,4]) = \left[3\frac{1}{4}, 3\frac{1}{2}\right]$ are disjoint, we may use Propositions 9.6 and 9.7 to obtain estimates for the dimensions of F.

For $x \in [2,4]$, $i = 1, 2$,

$$\frac{1}{16} \le |S_i'(x)| = \left|\frac{-1}{x^2}\right| \le \frac{1}{4}.$$

It follows from the mean-value theorem that, for $x, y \in [2,4]$, $i = 1, 2$,

$$\frac{1}{16}|x - y| \le |S_i(x) - S_i(y)| \le \frac{1}{4}|x - y|,$$

so in particular S_1 and S_2 are contractions. It follows from Propositions 9.6 and 9.7 that $t \le \dim_{\mathrm{H}} F \le s$, where

$$2(1/16)^t = 1 = 2(1/4)^s;$$

that is,

$$2^{1-4t} = 1 = 2^{1-2s}.$$

Thus $t = 1/4$ and $s = 1/2$; that is

$$1/4 \le \dim_{\mathrm{H}} F \le 1/2.$$

10.8 For a real number x and a positive integer Q, the set $\{rx \,(\mathrm{mod}1) : r = 0, 1, \ldots, Q\}$ contains $Q+1$ numbers in the interval $[0, 1]$, so two of these numbers will differ by $\le 1/Q$; thus there are integers $0 \le r \ne s \le Q$ such that $0 \le (s-r)x \,(\mathrm{mod}1) \le 1/Q$. Letting $q = |s-r|$ we have $0 < q \le Q$ and $-1/Q \le qx \,(\mathrm{mod}1) \le 1/Q$ so that $\|qx\| \le 1/Q$.

If x is rational, then $\|qx\| = 0$ is an integer for infinitely many q, and so $\|qx\| \le q^{-1}$ infinitely often. If x is irrational, then for each $K = 1, 2, \ldots$ we may find, by the above, positive integers $q_K \le K$ such that $0 < \|q_K x\| \le 1/K \le 1/q_K$. Since $\|q_K x\| \ne 0$ and $1/K \to 0$ as $K \to \infty$, there must be infinitely many distinct such q_K.

10.9 If $x^n - dy^n = 1$, then

$$d^{1/n} = \left(\frac{x^n - 1}{y^n}\right)^{1/n} = \frac{x}{y}\left(1 - \frac{1}{x^n}\right)^{1/n}.$$

Both d and y are positive integers so that $x > 1$ and hence $0 < 1 - 1/x^n < 1$. Thus

$$1 - \frac{1}{x^n} < \left(1 - \frac{1}{x^n}\right)^{1/n} < 1$$

and so

$$\left|d^{1/n} - \frac{x}{y}\right| = \frac{x}{y}\left|\left(1 - \frac{1}{x^n}\right)^{1/n} - 1\right| < \frac{x}{yx^n} = \frac{1}{yx^{n-1}}.$$

Since $x^n = 1 + dy^n$, and $d \ge 1$, we have $x > y$ and so

$$\left|d^{1/n} - \frac{x}{y}\right| < \frac{1}{y^n}.$$

If $x^n - dy^n = 1$ has infinitely many solutions (x, y), where x and y are positive integers, then, since it can have only one solution for each value of y, it follows that there are infinitely many positive integers y such that $|d^{1/n} - x/y| < 1/y^n$ for some integer x, that is, $d^{1/n}$ is n-well approximable.

10.10 For m and n integers, $(x, y) \in F$ if and only if $(x+m, y+n) \in F$. Thus $\dim_H F = \dim_H G$, where $G = F \cap ([0,1] \times [0,1])$.

For each integer q, let G_q denote the set of $(x, y) \in [0,1] \times [0,1]$ such that $\|qx\| \leq q^{1-\alpha}$ and $\|qy\| \leq q^{1-\alpha}$. Then G_q can be covered by the $(q+1)^2$ boxes of side $2/q^\alpha$ centered at the points $(p/q, p'/q)$, where $0 \leq p, p' \leq q$. We denote this collection of boxes by C_q. Clearly $G \subset \bigcup_{q=k}^{\infty} G_q$ and so G can be covered by $\bigcup_{q=k}^{\infty} \bigcup_{U \in C_q} U$. If k is sufficiently large to ensure that $2\sqrt{2}/k^\alpha \leq \delta$, then each of the boxes in C_q for $q \geq k$ has diameter at most δ and so

$$\mathcal{H}^s_\delta(G) \leq \sum_{q=k}^{\infty} (q+1)^2 (2\sqrt{2}/q^\alpha)^s.$$

If $s = 3/\alpha + \epsilon$ for some $\epsilon > 0$, then

$$\sum_{q=1}^{\infty} (q+1)^2 (2\sqrt{2}/q^\alpha)^s < 4^{1+s} \sum_{q=1}^{\infty} q^2/q^{\alpha s} = 4^{1+s} \sum_{q=1}^{\infty} 1/q^{1+\alpha\epsilon} < \infty$$

and so $\sum_{q=k}^{\infty} (q+1)^2 (2\sqrt{2}/q^\alpha)^s \to 0$ as $k \to \infty$. Since $k \to \infty$ as $\delta \to 0$, it follows that

$$\mathcal{H}^s(G) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(G) = 0,$$

if $s > 3/\alpha$. Thus $\dim_H F = \dim_H G \leq 3/\alpha$.

10.11 We use the sets of large intersection of Section 8.2. Let

$$\begin{aligned} F &= \{x : x \text{ is } \alpha\text{-well approximable }\} \\ &= \{x : \|qy\| \leq q^{1-\alpha} \text{ for infinitely many } q \ \}. \end{aligned}$$

Define $f_m^+, f_m^- : [0, \infty) \to \mathbb{R}$ by $f_m^+(x) = x^{1/2} - m$, $f_m^-(x) = -x^{1/2} - m$. By Proposition 10.4, $F \in \mathcal{C}^s[0, \infty)$ for all $s < 2/\alpha$. Since f_m^+, f_m^- are differentiable with derivative bounded away from 0 on $[m, M]$ for all $0 < m < M$, it follows from Proposition 8.8, by taking a countable union, that

$$\begin{aligned} F_m &= \{x : (x+m)^2 \text{ is } \alpha\text{-well approximable}\} \\ &= f_m^+(F \cap [0, \infty)) \cup f_m^-(F \cap [0, \infty)) \in \mathcal{C}^s(-\infty, \infty). \end{aligned}$$

By Corollary 8.7,

$$\dim_{\rm H}\{x : (x+m)^2 \text{ is } \alpha\text{-well approximable for all } m\}$$
$$= \dim_{\rm H}\left(\bigcap_{m=-\infty}^{\infty} F_m\right) \geq s$$

for all $s < 2/\alpha$.

On the other hand,

$$\dim_{\rm H}\{x : (x+m)^2 \text{ is } \alpha\text{-well approximable for all } m\}$$
$$\leq \dim_{\rm H}\{x : x^2 \text{ is } \alpha\text{-well approximable }\} = 2/\alpha$$

as in Proposition 10.4, so

$$\dim_{\rm H}\{x : (x+m)^2 \text{ is } \alpha\text{-well approximable for all } m\} = 2/\alpha.$$

Chapter 11

11.1 If f' is continuous on $[0, 1]$ then $f'([0, 1])$ is bounded. Thus there exists $0 < c < \infty$ such that $|f'(t)| \leq c$, for each $t \in [0, 1]$. It follows from the mean-value theorem that, for $0 \leq t, u \leq 1$,

$$|f(t) - f(u)| \leq c|t-u|.$$

Thus (11.2) is satisfied with $s = 1$ and so it follows from Corollary 11.2(a) that $\mathcal{H}^1(\text{graph} f) < \infty$.

The graph of f is a continuous curve joining the points in the plane $(0, f(0))$ and $(1, f(1))$. The projection of this curve onto the x-axis is the interval $[0, 1]$, so $\mathcal{H}^1(\text{graph} f) \geq \mathcal{H}^1([0, 1]) = 1$, by Proposition 2.2 and (6.1). Thus $0 < \mathcal{H}^1(\text{graph} f) < \infty$.

To show that the graph is a regular set, we show that the graph is a rectifiable curve and apply Lemma 5.5. For $0 = t_0 < t_1 < \ldots < t_m = 1$ we have polygonal approximations to the length of the graph given by

$$\sum_{i=1}^{m} |(t_i, f(t_i)) - (t_{i-1}, f(t_{i-1}))|$$
$$= \sum_{i=1}^{m} \left((t_i - t_{i-1})^2 + (f(t_i) - f(t_{i-1}))^2\right)^{1/2}$$

$$\leq \sum_{i=1}^{m}\left((t_i - t_{i-1})^2 + c^2(t_i - t_{i-1})^2\right)^{1/2} = (1+c^2)^{1/2}\sum_{i=1}^{m}|t_i - t_{i-1}|$$
$$= (1+c^2)^{1/2} < \infty.$$

Hence the supremum of the lengths of the polygonal approximations to the graph is finite, so graph$f$ is a rectifiable curve in the plane, and so by Lemma 5.5 is a regular 1-set.

11.2 Assume that $|f(t) - f(u)| \leq c|t-u|$ for $0 \leq t, u \leq 1$. Define ψ : graph$g \to$ graph$(f+g)$ by $\psi(t, g(t)) = (t, f(t) + g(t))$. Then

$$\begin{aligned}
|\psi(t, g(t)) - \psi(u, g(u))|^2 &= |(t, f(t)+g(t)) - (u, f(u)+g(u))|^2\\
&= |t-u|^2 + |f(t) - f(u) + g(t) - g(u)|^2\\
&\leq |t-u|^2 + 2|f(t) - f(u)|^2 + 2|g(t) - g(u)|^2\\
&\leq c|t-u|^2 + 2|g(t) - g(u)|^2\\
&\leq c_1(|t-u|^2 + |g(t) - g(u)|^2)\\
&= |(t, g(t)) - (u, g(u))|^2.
\end{aligned}$$

Thus ψ is Lipschitz. On the other hand,

$$\begin{aligned}
&|\psi(t, g(t)) - \psi(u, g(u))| = |(t, f(t)+g(t)) - (u, f(u)+g(u))|\\
&\geq \max\{|t-u|, |f(t) - f(u) + g(t) - g(u)|\}\\
&\geq \max\{|t-u|, |g(t) - g(u)| - |f(t) - f(u)|\}\\
&\geq \max\{|t-u|, |g(t) - g(u)| - c|t-u|\}\\
&\geq ((c+1)|t-u| + (|g(t) - g(u)| - c|t-u|))\,/(c+2)\\
&= (|t-u| + |g(t) - g(u)|)/(c+2)\\
&\geq |(t, g(t)) - (u, g(u))|/(c+2),
\end{aligned}$$

using that $\max\{a, b\} \geq ((c+1)a + b)/(c+2)$. Thus ψ is bi-Lipschitz, so that $\dim_{\mathrm{H}}$graph$g = \dim_{\mathrm{H}}$graph$(f+g)$, with similar equality for box dimensions.

11.3 If the box dimensions of graph$f$ and graph$g$ exist, then it follows from Proposition 11.1 that

$$\dim_{\mathrm{B}}\mathrm{graph} f = \lim_{\delta\to 0}\frac{\log N_\delta}{-\log\delta} = \lim_{\delta\to 0}\frac{-\log\delta + \log\sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta]}{-\log\delta}$$
$$= 1 + \lim_{\delta\to 0}\frac{\log\sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta]}{-\log\delta}$$

and

$$\dim_{\mathrm{B}} \mathrm{graph}\, g = 1 + \lim_{\delta \to 0} \frac{\log \sum_{i=0}^{m-1} R_g[i\delta, (i+1)\delta]}{-\log \delta}.$$

If $\dim_{\mathrm{B}} \mathrm{graph}\, f = \dim_{\mathrm{B}} \mathrm{graph} g + 2\epsilon$, for some $\epsilon > 0$, then it follows that there exists $\delta_0 > 0$ such that, for all $\delta < \delta_0$,

$$\frac{\log \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta]}{-\log \delta} > \frac{\log \sum_{i=0}^{m-1} R_g[i\delta, (i+1)\delta]}{-\log \delta} + \epsilon$$

and hence

$$\begin{aligned}\sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta] &> e^{(-\log \delta)\epsilon} \sum_{i=0}^{m-1} R_g[i\delta, (i+1)\delta] \\ &= (1/\delta)^{\epsilon} \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta].\end{aligned}$$

Now, for any interval $[t_1, t_2] \subset [0, 1]$,

$$\begin{aligned}R_f[t_1, t_2] - R_g[t_1, t_2] \leq R_{f+g}[t_1, t_2] &= \sup_{t_1 < t, u < t_2} |f(t) + g(t) - f(t) - g(t)| \\ &\leq R_f[t_1, t_2] + R_g[t_1, t_2].\end{aligned}$$

So, for $\delta < \delta_0$,

$$\begin{aligned}(1 - \delta^{\epsilon}) \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta] &\leq \sum_{i=0}^{m-1} R_{f+g}[i\delta, (i+1)\delta] \\ &\leq (1 + \delta^{\epsilon}) \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta]\end{aligned}$$

and hence

$$\begin{aligned}\frac{\log \sum_{i=0}^{m-1} R_{f+g}[i\delta, (i+1)\delta]}{-\log \delta} &\leq \frac{\log \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta] + \log(1 + \delta^{\epsilon})}{-\log \delta} \\ &\leq \frac{\log \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta] + \delta^{\epsilon}}{-\log \delta}.\end{aligned}$$

Similarly,

$$\frac{\log \sum_{i=0}^{m-1} R_{f+g}[i\delta, (i+1)\delta]}{-\log \delta} \geq \frac{\log \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta] - \delta^{\epsilon}}{-\log \delta}.$$

Thus

$$\lim_{\delta\to 0} \frac{\log \sum_{i=0}^{m-1} R_{f+g}[i\delta, (i+1)\delta]}{-\log\delta} = \lim_{\delta\to 0} \frac{\log \sum_{i=0}^{m-1} R_f[i\delta, (i+1)\delta]}{-\log\delta}$$

and hence $\dim_{\mathrm{B}}\mathrm{graph}(f+g) = \dim_{\mathrm{B}}\mathrm{graph}\, f$.

To see why we must require $\dim_{\mathrm{B}}\mathrm{graph}\, f$ and $\dim_{\mathrm{B}}\mathrm{graph}\, g$ to be unequal, consider the case when $\dim_{\mathrm{B}}\mathrm{graph}\, f > 1$ and $g(t) = -f(t)$ so that graph $f+g$ is a straight line and hence has box dimension one.

11.4 Given that (11.3) holds with $1 < s < 2$, we have that, for all $t \in [0,1]$ and $0 < \delta \le \delta_0$, there exists u with $|t-u| \le \delta$ such that

$$\left|\frac{f(u)-f(t)}{t-u}\right| \ge \frac{c\delta^{2-s}}{|t-u|} \ge \frac{c|t-u|^{2-s}}{|t-u|} = c|t-u|^{1-s}.$$

Hence

$$\overline{\lim_{u\to t}}\left|\frac{f(u)-f(t)}{t-u}\right| \ge \overline{\lim_{u\to t}}\, c|t-u|^{1-s} = \infty$$

and so the derivative $f'(t)$ at t does not exist.

This condition (11.3) is satisfied by the Weierstrass function, see the penultimate line of the Calculation of Example 11.3, so the Weierstrass function is nowhere differentiable.

For the self-affine functions f of Example 11.4, note that from (11.9) there is a number $0 < \epsilon < 1$ such that $m^{-1+\epsilon} \le c_i$ for all i. Thus, from the calculation of Example 11.4, $dm^{(-1+\epsilon)k} \le dc_{i_1}\cdots c_{i_k} \le R_f[I_{i_1,\dots,i_k}]$ for each interval $I_{i_1,\dots,i_k}$, this interval having length m^{-k}. Thus given $t \in [0,1]$ and $0 < \delta < 1$ we may find an interval $I \equiv I_{i_1,\dots,i_k}$ containing t and with length $|I| = m^{-k} \le \delta < m^{-k+1}$. There are points $u_1, u_2 \in I$ with

$$|f(u_2) - f(u_1)| = R_f[I_{i_1,\dots,i_k}] \ge dm^{(-1+\epsilon)k} \ge (\delta m^{-1})^{1-\epsilon},$$

so since either $|f(t) - f(u_1)| \ge \frac{1}{2}|f(u_2) - f(u_1)|$ or $|f(t) - f(u_2)| \ge \frac{1}{2}|f(u_2) - f(u_1)|$, we conclude that there is u with $|t-u| \le \delta$ such that $|f(t) - f(u)| \ge \frac{1}{2}m^{\epsilon-1}\delta^{1-\epsilon}$. This is condition (11.3) with $s = 1+\epsilon$, so by the first part of the question, the self-affine function f is nowhere differentiable.

11.5 The calculation is similar to that for Example 11.3 and the solution to Exercise 11.5. Given $0 < h < \lambda^{-1}$, let N be the integer such that

$$\lambda^{-(N+1)} \le h < \lambda^{-N}.$$

Then

$$\begin{aligned}|f(t+h) - f(t)| \le \sum_{k=1}^{N} \lambda^{(s-2)k} |\sin(\lambda^k(t+h)+\theta_k) - \sin(\lambda^k t + \theta_k)| \\ + \sum_{k=N+1}^{\infty} \lambda^{(s-2)k} |\sin(\lambda^k(t+h)+\theta_k) - \sin(\lambda^k t + \theta_k)|.\end{aligned}$$

Let $g(t) = \sin(\lambda^k t + \theta_k)$, then

$$|g'(t)| = \lambda^k |\cos(\lambda^k t + \theta_k))| \le \lambda^k$$

and so, by the mean-value theorem,

$$\sum_{k=1}^{N} \lambda^{(s-2)k} |\sin(\lambda^k(t+h)+\theta_k) - \sin(\lambda^k t + \theta_k)| \le \sum_{k=1}^{N} \lambda^{(s-2)k}\lambda^k h.$$

Since $|\sin t| \le 1$ for all real values of t, we have

$$\sum_{k=N+1}^{\infty} \lambda^{(s-2)k} |\sin(\lambda^k(t+h)+\theta_k) - \sin(\lambda^k t + \theta_k)| \le \sum_{k=N+1}^{\infty} 2\lambda^{(s-2)k}.$$

So,

$$\begin{aligned}|f(t+h) - f(t)| &\le \sum_{k=1}^{N} \lambda^{(s-2)k}\lambda^k h + \sum_{k=N+1}^{\infty} 2\lambda^{(s-2)k} \\ &\le \frac{h\lambda^{(s-1)N}}{1-\lambda^{1-s}} + \frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}}.\end{aligned}$$

Since $\lambda^{-(N+1)} \le h < \lambda^{-N}$, it follows that

$$|f(t+h) - f(t)| \le \frac{hh^{1-s}}{1-\lambda^{1-s}} + \frac{2h^{2-s}}{1-\lambda^{s-2}} \le ch^{2-s},$$

where c is independent of h. It now follows from Corollary 11.2(a) that $\overline{\dim}_B \operatorname{graph} f \le s$.

Similar arguments show that,

$$|f(t+h)-f(t)-\lambda^{(s-2)N}(\sin(\lambda^N(t+h)+\theta_k)-\sin(\lambda^N t+\theta_k))|$$
$$\leq \sum_{k=1}^{N-1}\lambda^{(s-2)k}\lambda^k h+\sum_{k=N+1}^{\infty}2\lambda^{(s-2)k}$$
$$\leq \frac{h\lambda^{(s-1)(N-1)}}{1-\lambda^{1-s}}+\frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}}$$
$$\leq \frac{\lambda^{(s-2)N-s+1}}{1-\lambda^{1-s}}+\frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}},$$

if $\lambda^{-(N+1)} \leq h < \lambda^{-N}$.

We now observe that, since sin is a periodic function with period 2π and is strictly increasing on $(-\pi/2, \pi/2)$ and strictly decreasing on $(\pi/2, 3\pi/2)$, then there exists $c > 0$ such that, for each $T \in \mathbb{R}$ we may choose H with $1/2 \leq H < 1$ such that $|\sin(T+H)-\sin T| > c$.

We note that, if $h = \lambda^{-N}$, then

$$\lambda^N(t+h)+\theta_k-(\lambda^N t+\theta_k)=\lambda^N h=1,$$

and, if $h = \lambda^{-(N+1)}$, then

$$\lambda^N(t+h)+\theta_k-(\lambda^N t+\theta_k)=\lambda^N h=\lambda^{-1}<1/2,$$

provided that $\lambda > 10$. Thus, if $\lambda > 10$, then, for each $t \in (0, 1)$ and each N, we may choose h with $\lambda^{-(N+1)} \leq h < \lambda^{-N}$ such that $|\sin(\lambda^N(t+h)+\theta_k)-\sin(\lambda^N(t+h)+\theta_k)| > c$.

If λ is sufficiently large, then

$$\frac{\lambda^{(s-2)N-s+1}}{1-\lambda^{1-s}}+\frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}}<\frac{c\lambda^{(s-2)N}}{2},$$

for all N, and so, for each $t \in (0, 1)$ and each N, we may choose h with $\lambda^{(-N+1)} \leq h < \lambda^{-N} = \delta$ such that

$$|f(t+h)-f(t)| \geq c\lambda^{(s-2)N}-c\lambda^{(s-2)N}/2 \geq c\lambda^{(s-2)N}/2 \geq c\delta^{2-s}/2.$$

It now follows from Corollary 11.2(b) that $\underline{\dim}_B \operatorname{graph} f \geq s$.

11.6 The calculation is similar to that for Example 11.3. Given $0 < h < \lambda^{-1}$, let N be the integer such that

$$\lambda^{-(N+1)} \leq h < \lambda^{-N}.$$

Then

$$|f(t+h)-f(t)| \le \sum_{k=1}^{N} \lambda^{(s-2)k}|g(\lambda^k(t+h))-g(\lambda^k t)| + \sum_{k=N+1}^{\infty} \lambda^{(s-2)k}|g(\lambda^k(t+h))-g(\lambda^k t)|.$$

We note from its zig-zag form that g is a Lipschitz function with

$$|g(t_1)-g(t_2)| \le |t_1-t_2| \text{ for all } t_1, t_2 \in \mathbb{R}$$

and so

$$\sum_{k=1}^{N} \lambda^{(s-2)k}|g(\lambda^k(t+h))-g(\lambda^k t)| \le \sum_{k=1}^{N} \lambda^{(s-2)k}\lambda^k h.$$

Since $|g(t)| \le 1$ for all real values of t, we have

$$\sum_{k=N+1}^{\infty} \lambda^{(s-2)k}|g(\lambda^k(t+h))-g(\lambda^k t)| \le \sum_{k=N+1}^{\infty} 2\lambda^{(s-2)k}.$$

Thus

$$|f(t+h)-f(t)| \le \sum_{k=1}^{N} \lambda^{(s-2)k}\lambda^k h + \sum_{k=N+1}^{\infty} 2\lambda^{(s-2)k} \le ch^{2-s},$$

where c is independent of h. It now follows from Corollary 11.2(a) that $\overline{\dim}_{\mathrm{B}}\mathrm{graph} f \le s$.

In the same way,

$$|f(t+h)-f(t)-\lambda^{(s-2)N}(g(\lambda^N(t+h))-g(\lambda^N t))| \le \frac{\lambda^{(s-2)N-s+1}}{1-\lambda^{1-s}} + \frac{2\lambda^{(s-2)(N+1)}}{1-\lambda^{s-2}},$$

if $\lambda^{-(N+1)} \le h < \lambda^{-N}$.

We now observe that, since g is a periodic function with period 4 and is strictly increasing on $(-1, 1)$ and strictly decreasing on $(1, 3)$, then there exists $c > 0$ such that, for each $t \in (0, 1)$ and each positive integer N, we may choose h with $1/2 \le \lambda^N h < 1$ such that $|g(\lambda^N(t+h)) - g(\lambda^N t)| > c$. If $\lambda > 2$, then this implies that, for each $t \in (0, 1)$ and each positive integer N, we may choose h with $\lambda^{(-N+1)} \le h < \lambda^{-N}$ such that $|g(\lambda^N(t+h)) - g(\lambda^N t)| > c$. If λ is sufficiently large, then the right-hand side of the last displayed inequality above will be less than $c\lambda^{(s-2)N}/2$

for all N. It then follows from the same inequality that, for each $t \in (0,1)$ and each N, we may choose h with $\lambda^{(-N+1)} \le h < \lambda^{-N} = \delta$ such that

$$|f(t+h) - f(t)| \ge \lambda^{(s-2)N}/2 \ge c\delta^{2-s/2}.$$

It now follows from Corollary 11.2(b) that $\underline{\dim}_{\mathrm{B}}\mathrm{graph} f \ge s$.

11.7 From Proposition 2.3 and (11.2) we see that $\dim_{\mathrm{H}} f(F) \le \min\{1, \dim_{\mathrm{H}} F/(2-s)\}$.

More interesting is the dimension of the subset of graph$f$ given by $E = \{(t, f(t)) : t \in F\}$. Suppose that $F \subset [0,1]$ intersects $N_\delta(F)$ of the δ-mesh intervals. For such an interval I the maximum range $R[I] \le c\delta^{2-s}$ by (11.2). Thus the portion of E above the interval I may be covered by $c\delta^{2-s}\delta^{-1} + 1 = c\delta^{1-s} + 1$ squares of side δ, so the number of squares of side δ needed to cover E is $N_\delta(E) \le N_\delta(F)(c\delta^{1-s} + 1)$. Hence

$$\frac{\log N_\delta(E)}{-\log\delta} \le \frac{\log N_\delta(F)}{-\log\delta} + \frac{\log(c\delta^{1-s}+1)}{-\log\delta}$$

so taking lower and upper limits as $\delta \to 0$ gives $\underline{\dim}_{\mathrm{B}} E \le \underline{\dim}_{\mathrm{B}} F + (s-1)$ and $\overline{\dim}_{\mathrm{B}} E \le \overline{\dim}_{\mathrm{B}} F + (s-1)$.

11.8 Define a measure μ on graph$f$ by $\mu(A) = \mathcal{L}^1\{t \in [0,1] : (t, f(t)) \in A\}$ for $A \subset \mathbb{R}^2$, so that for measurable $g : \mathbb{R}^2 \to \mathbb{R}$ we have $\int g(x)d\mu(x) = \int_0^1 g(t, f(t))dt$. Then

$$\begin{aligned}\int\int \frac{d\mu(x)d\mu(y)}{|x-y|^s} &= \int\int |(t, f(t)) - (u, f(u))|^{-s} dt du \\ &= \int\int [|t-u|^2 + |f(t) - f(u)|^2]^{-s/2} dt du < \infty\end{aligned}$$

by the given condition. Since μ is supported by graph$f$, it follows from Theorem 4.13(a) that $\dim_{\mathrm{H}}(\mathrm{graph} f) \ge s$.

11.9 Let S be a δ-mesh square of D. Then the maximum range over the square $R[S] = \sup_{t,u\in S}|f(t) - f(u)| \le c(\delta\sqrt{2})^{3-s}$ where $2 \le s < 3$. Thus the portion of the surface $F = \{(t, f(t)) : t \in D\}$ above the square S may be covered by $c(\delta\sqrt{2})^{3-s}\delta^{-1} + 1 = c2^{(3-s)/2}\delta^{2-s} + 1$ cubes of side δ, so the number of mesh cubes of side δ needed to cover F is $N_\delta(F) \le (c2^{(3-s)/2}\delta^{2-s} + 1)(\delta^{-1} + 1)^2 \le 20c\delta^{-s}$ for small δ. Hence

$$\frac{\log N_\delta(F)}{-\log\delta} \le \frac{\log(20c\delta^{-s})}{-\log\delta} = \frac{\log 20c - s\log\delta}{-\log\delta}$$

so taking upper limits as $\delta \to 0$ gives $\underline{\dim}_{\mathrm{B}} F \le \overline{\dim}_{\mathrm{B}} F \le s$.

We now give a surface analogue to Corollary 11.2(b). Suppose that there are numbers $c > 0$, $\delta_0 > 0$ and $2 \le s < 3$ with the property: for each $t \in S$ and $0 < \delta \le \delta_0$ there exists u such that $|t - u| \le \delta$ and

$$|f(t) - f(u)| \ge c\delta^{3-s}.$$

Then $s \le \underline{\dim}_B F$ where $F = \{(t, f(t)) : t \in S\}$.

To prove this, we note that for a square S of side δ the maximum range $R[S] = \sup_{t,u\in S} |f(t) - f(u)| \ge c(\delta/2)^{3-s}$, so at least $c(\delta/2)^{3-s}\delta^{-1} = c2^{s-3}\delta^{2-s}$ cubes of side δ are needed to cover the portion of the surface F above a square S of side δ. Thus the number of mesh cubes of side δ needed to cover F is at least $(c2^{s-3}\delta^{2-s})(\delta^{-2}) \ge c2^{s-3}\delta^{-s}$. Hence

$$\frac{\log N_\delta(F)}{-\log\delta} \ge \frac{\log(c2^{s-3}\delta^{-s})}{-\log\delta} = \frac{\log(c2^{s-3}) - s\log\delta}{-\log\delta},$$

so taking lower limits as $\delta \to 0$ gives $\underline{\dim}_B F \ge s$.

11.10 The transformations S_1 and S_2 are of the form (11.8) with

$$m = 2,\ a_1 = 1/4,\ b_1 = 0,\ c_1 = 5/6,\ a_2 = -1/4,\ b_2 = 1/4,\ c_2 = 5/6.$$

To verify that the attractor F of S_1 and S_2 is the graph of a continuous function, we must check that conditions (11.9) and (11.10) are satisfied with $S_1(p_1)$, $S_2(p_1)$ and p_2 not all collinear. We begin by noting that

$$1/m = 1/2 < 5/6 = c_1 = c_2$$

so that (11.9) is satisfied. Also, $p_1 = (0, b_1/(1 - c_1)) = (0, 0)$ and $p_2 = (1, (a_2 + b_2)/(1 - c_2)) = (1, 0)$, so that $S_1(p_1) = p_1 = (0, 0)$, $S_2(p_1) = (1/2, 1/4)$ and $p_2 = (1, 0)$ are not all collinear.

We must now check that the fixed points p_1 and p_2 of S_1 and S_2 satisfy $S_1(p_2) = S_2(p_1)$. We have $S_2(1, 0) = (1/2, 1/4)$, so $S_1(p_2) = S_2(p_1) = (1/2, 1/4)$. Thus F is the graph of a self-affine continuous fractal curve.

We calculate that $S_1(q) = (1/4, 27/8)$ and $S_2(q) = (3/4, 5/2)$ so that E_2 may be sketched.

The box dimension of F is given by the formula in Example 11.4:

$$\dim_B F = 1 + \frac{\log(c_1 + c_2)}{\log m} = 1 + \frac{\log(5/3)}{\log 2} = 1.737$$

to three decimal places.

11.11 The transformations S_1, S_2 and S_3 are of the form (11.8) with

$$m = 3,\ a_1 = 1/3,\ b_1 = 0,\ c_1 = 1/2,\ a_2 = -2/3,\ b_2 = 1/3,\ c_2 = 1/2$$

$$a_3 = 1/3,\ b_3 = -1/3,\ c_3 = 1/2.$$

To verify that the attractor F of S_1, S_2 and S_3 is the graph of a continuous function, we must check that conditions (11.9) and (11.10) are satisfied with, say, $S_1(p_1)$, $S_2(p_1)$ and p_2 not all collinear. We begin by noting that

$$1/m = 1/3 < 1/2 = c_1 = c_2 = c_3$$

so that (11.9) is satisfied. Also, $p_1 = (0, b_1/(1 - c_1)) = (0, 0)$ and $p_3 = (1, (a_3 + b_3)/(1 - c_3)) = (1, 0)$, so that $S_1(p_1) = p_1 = (0, 0)$, $S_2(p_1) = S_2(0, 0) = (1/3, 1/3)$ and $p_2 = (1, 0)$ are not all collinear.

Now note that $S_1(p_3) = (1/3, 1/3) = S_2(p_1)$ and $S_2(p_3) = (2/3, -1/3) = S_3(p_1)$, so (11.10) is satisfied and so F is the graph of a self-affine continuous fractal curve. The points on the polygon E_2 may be calculated as: (0,0), (1/9,5/18), (2/9,1/18), (1/3,1/3), (4/9,5/18), $(5/9, -5/18)$, $(2/3, -1/3)$, $(7/9, -1/18)$, $(8/9, -1/18)$, (1,0).

The box dimension of F is given by the formula in Example 11.4:

$$\dim_{\mathrm{B}} F = 1 + \frac{\log(c_1 + c_2 + c_3)}{\log m} = 1 + \frac{\log(3/2)}{\log 3} = 1.369$$

to three decimal places.

11.12 Let $f : [0, 1] \to \mathbb{R}$ be the Weierstrass function. The calculation in Example 11.3 shows that there is a constant c such that $|f(t + h) - f(t)| \le ch^{2-s}$ if $0 < h \le 1$, so

$$\frac{1}{2T}\int_{-T}^{T}(f(t + h) - f(t))^2 dt \le c^2 h^{4-2s}.$$

On the other hand, the end of the calculation in Example 11.3 shows that for some constant c_1 there exist numbers $h > 0$ arbitrarily close to 0 such that $|f(t + h) - f(t)| \ge c_1 h^{2-s}$ for all t, so for such h

$$\frac{1}{2T}\int_{-T}^{T}(f(t + h) - f(t))^2 dt \ge c_1 h^{4-2s}.$$

By (11.13)

$$\log(C(0) - C(h)) = \log\left(\frac{1}{2}\overline{\lim}_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(f(t + h) - f(t))^2 dt\right).$$

Hence

$$\overline{\lim}_{h\to 0}\frac{\log(C(0) - C(h))}{\log h} = 4 - 2s.$$

Chapter 12

12.1 In a similar way to Proposition 12.2, let F be an irregular 1-set in the unit square such that $\mathrm{proj}_0 F$ contains the interval $[0, 1]$ of the x-axis. Define a mapping $\psi : [0, 1] \times [0, 1] \to \mathbb{R}^2$ by $\psi(x, y) = (x(1 + y^2)^{1/2}, y)$. It is easy to see that ψ is bi-Lipschitz and continuously differentiable, so $\psi(F)$ is an irregular 1-set. (Such maps preserve irregularity, see for example, Exercise 5.2.) For all $0 \le d \le 1$, there is a point $(d, b) \in F$ for some b, so there is a point $(d(1 + b^2)^{1/2}, b) \in \psi(F)$, that is a point $(a, b) \in \psi(F)$ with $a = d(1 + b^2)^{1/2}$, that is with $a/(1 + b^2)^{1/2} = d$.

By Proposition 12.1(b) the line set $L(\psi(F))$ has area 0. However, the line $y = a + bx$ is at perpendicular distance $a/(1 + b^2)^{1/2}$ from the origin, so since there are points in $\psi(F)$ for which this expression takes all values in $[0, 1]$, the set $L(\psi(F))$ contains lines at all perpendicular distances between 0 and 1 from the origin, as required.

12.2 The mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(r, \theta) = (1/r, \theta)$ transforms a line at perpendicular distance R from the origin to a circle of radius $1/R$ through the origin. Thus, taking $E = L(\psi(F))$ to be the set of the last exercise, $\phi(E)$ contains a circle of radius $1/R$ through the origin for all $0 < R \le 1$. Clearly, ϕ maps sets of area 0 to sets of area 0, so $\phi(E)$ is a set of area 0 containing a circle of every radius ≥ 1. Taking a union $\bigcup_{n=1}^{\infty} \frac{1}{n}\phi(E)$, where $\frac{1}{n}\phi(E)$ is the set $\phi(E)$ scaled about the origin by a factor $\frac{1}{n}$, we get a set of zero area containing a circle of every positive radius.

12.3 This is a variation of Proposition 12.2. Let F be any irregular 1-set such that the projection onto the x-axis contains the unit interval $[0, 1]$. By Proposition 12.1(b) the line set $L(F)$ has area 0. The line $L(a, b)$ given by $y = a + bx$ cuts the y axis at $a = \mathrm{proj}_0(a, b)$, so since $[0, 1] \subset \mathrm{proj}_0 F$, the line set $L(F)$ contains lines cutting the y-axis at every point of the interval $[0, 1]$. Taking a countable union of translates $\bigcup_{n=-\infty}^{\infty}(L(F) + (0, n))$ gives a set of area 0 containing a line cutting the y-axis at each of its point, which is essentially the required set.

12.4 This is an extension of the second part of Proposition 12.2. Writing $L(a, b)$ for the set of points in the plane on the line $y = a + bx$, let $E = \{(a, b) : L(a, b) \subset F\}$ so that $F \supset L(E)$. Then, since F contains a line in every direction θ for $\theta \in A$, $\mathrm{proj}_{\pi/2} E \supset \{\tan\theta : \theta \in A\}$. Thus $\dim_{\mathrm{H}} E \ge \dim_{\mathrm{H}} \mathrm{proj}_{\pi/2} E \ge \dim_{\mathrm{H}} A$. By Proposition 12.1(a),

$$\begin{aligned}\dim_{\mathrm{H}} F \ge \dim_{\mathrm{H}} L(E) &\ge \min\{2, 1 + \dim_{\mathrm{H}} E\}\\ &\ge \min\{2, 1 + \dim_{\mathrm{H}} A\} = 1 + \dim_{\mathrm{H}} A.\end{aligned}$$

12.5 Let A be a Borel subset of $\mathbb{R}^2$ of area a. For each $\theta \in [0,\pi)$ define A_θ to be the set $(1+c^2)^{-1}(A_\theta \cap L_c)$, where $c = \tan\theta$, L_c is the line $x = c$, and we have scaled the set $A_\theta \cap L_c$ by a factor $(1+c^2)^{-1}$. By Theorem 6.9 there is a compact set $F \subset \mathbb{R}^2$ such that $\mathrm{proj}_\theta F \supset A_\theta$ for all θ and $\mathcal{L}^1(\mathrm{proj}_\theta F) = \mathcal{L}^1(A_\theta)$ for almost all θ. By duality, writing $L(F)$ for the line set of F, we have $L(F) \cap L_c$ is congruent to $\mathrm{proj}_\theta F$. It follows that for all c we have $L(F) \cap L_c \supset A \cap L_c$ with $\mathcal{L}^1(L(F) \cap L_c) = \mathcal{L}^1(A \cap L_c)$ for almost all c, so we have $L(F) \supset A$ and on integrating $\mathcal{L}^2(L(F)) = \mathcal{L}^2(A)$, as required.

12.6 Note that if μ is supported by F then $f(z) = \int_F (z-w)^{-1} d\mu(w)$ is analytic at $z \in \mathbb{C} \setminus F$. Thus for F to be removable, there would have to be an analytic function $\tilde{f}(z)$ with $f(z) = \tilde{f}(z)$ for $z \in \mathbb{C}\setminus F$. In particular, by Cauchy's identity, for every contour C we would require

$$\int_C \tilde{f}(z)dz = \int_C f(z)dz = \int_C \int_F \frac{d\mu(w)}{(z-w)} dz = \int_F \int_C \frac{dz}{(z-w)} d\mu(w)$$
$$= \int_F 2\pi i\, d\mu(w) = 2\pi i \mu(F) > 0,$$

provide C encloses F. By Cauchy's theorem, $\tilde{f}(z)$ cannot be analytic on any domain containing C, so F is not removable.

If $1 < \dim_{\mathrm{H}} F$, Theorem 4.13(b) gives a mass distribution μ on F and a constant M such that $\int |z-w|^{-1} d\mu(w) \le M$ for all $z \in \mathbb{R}^2$. Identifying $\mathbb{R}$ with $\mathbb{C}$ gives

$$\left| \int \frac{d\mu(w)}{(z-w)} \right| \le \int \frac{d\mu(w)}{|z-w|} \le M,$$

so by the first part, F is not removable.

12.7 Let $F = \{x_1, \ldots, x_k\}$ be a finite subset of $\mathbb{C}$. Let V be an open domain containing F and let f be a bounded analytic function on $V \setminus F$, say $|f(z)| \le M$. Let C be a contour in V enclosing F. Given $\epsilon > 0$ let $C_1, \ldots, C_k$ be contours with centres $x_1, \ldots, x_k$ and radii ϵ; we may assume that ϵ is small enough so that the contours are disjoint. By Cauchy's integral formula,

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)dw}{z-w} - \sum_j \frac{1}{2\pi i} \int_{C_j} \frac{f(w)dw}{z-w}$$

for z inside C but outside all of the C_j (to see this make cuts to join the contours C_j to C to form a single contour). Thus

$$\left| f(z) - \frac{1}{2\pi i} \int_C \frac{f(w)dw}{z-w} \right| \le \sum_j \frac{1}{2\pi} \left| \int_{C_j} \frac{f(w)dw}{z-w} \right| \le \sum_j \frac{1}{2\pi} \frac{M 2\pi\epsilon}{d(z, C_j)}$$

where $d(z, C_j)$ is the distance from z to the contour C_j. Letting $\epsilon \to 0$ gives that

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(w)dw}{z - w}.$$

But this defines an analytic function throughout the interior of V including at the points x_j, so this formula defines the required analytic extension of f.

Note that a slight modification of this proof shows that any compact set F with $\mathcal{H}^1(F) = 0$ is removable.

To see that the unit circle F is not removable, consider the function on $\mathbb{C}\backslash F$ given by $f(z) = 2$ if $|z| < 1$ and $f(z) = 1/z$ if $|z| > 1$. Then f is analytic and bounded on $\mathbb{C}\backslash F$, but clearly has no analytic extension to any region containing F since such an extension would be discontinuous on F. Thus F is not removable.

12.8 Define a function g by taking $g(t)$ to be the point $x \in \operatorname{graph} f$ such that $\operatorname{graph} f$ has a line of support with slope t at x; then g is defined on some maximal sub-interval I of $(-\pi/2, \pi/2)$. For each $x \in \operatorname{graph} f$, we have that $g^{-1}(x)$ is a closed interval, which is a single point if and only if $\operatorname{graph} f$ has a unique tangent at x, that is if and only if f is differentiable at x. For each k define the set $A_k = \{x : g^{-1}(x)$ is an interval of length $\geq 1/k\}$. Then since the intervals $g^{-1}(x)$ and $g^{-1}(y)$ are disjoint if $x \neq y$ and $g^{-1}(\mathbb{R}) = I \subset (-\pi/2, \pi/2)$, it follows that A_k contains at most π/k points, so in particular A_k is finite. The set of x at which f is not differentiable is $\bigcup_{k=1}^{\infty} A_k$ which is therefore either finite or countable.

12.9 The function $f : \mathbb{R}^2 \to \mathbb{R}$ given in coordinate form by $f(x, y) = |x|$ is easily seen to be convex, with set of non-differentiability the y-axis, which has Hausdorff dimension 1.

The Weierstrass function $f : \mathbb{R} \to \mathbb{R}$ given by (11.4) is continuous but nowhere differentiable, see Exercise 11.4, so the function $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = f(x)$ is nowhere differentiable on the plane, i.e. the non-differentiability set has Hausdorff dimension 2.

12.10 Let G be a subgroup of $(\mathbb{R}, +)$. We have two cases:

(i) For all $\epsilon > 0$ there exists $x \in G \cap (0, \epsilon)$. Then for all $y \in \mathbb{R}$ and $\epsilon > 0$ we have $nx \in (y - \epsilon, y + \epsilon)$ for some integer n, and also $nx \in G$, as an n-fold sum of x or $-x$. Hence the set of elements of G is dense in $\mathbb{R}$. Thus for every interval $[a, b]$ we have $\dim_B(G \cap [a, b]) = \dim_B(\overline{G \cap [a, b]}) = \dim_B [a, b] = 1$, using proposition 3.4. (In any meaningful sense $\dim_B G = 1$ also, though we have not defined box dimension for unbounded sets.)

(ii) There exists $\epsilon > 0$ such that $G \cap (0, 2\epsilon) = \emptyset$. Then for all $y \in \mathbb{R}$ the interval $(y - \epsilon, y + \epsilon)$ contains at most one element of G (if it contained two such elements their difference would be in $(0, 2\epsilon)$. It follows that for every interval $[a, b]$ the set $G \cap [a, b]$ is finite, so that $\dim_B(G \cap [a, b]) = 0$, using Proposition 3.4. (In any meaningful sense $\dim_B G = 0$ also, though we have not defined box dimension for unbounded sets.)

12.11 For $0 < t < 2$ let F be the set in Example 12.4 with $s = t/2$, so that $F = \bigcup_{r=1}^{\infty} F_r$ is a subgroup of $\mathbb{R}$ with $\dim_H F = s$ and $\dim_H F_r = \dim_B F_r = s$ (from Example 4.7). Consider $F \times F$. Then $F \times F$ is a group, with $(0, 0) \in F \times F$, with $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2) \in F \times F$ whenever $x_1, y_1, x_2, y_2 \in F$, using the group properties of F, and with $-(x, y) = (-x, -y) \in F \times F$ whenever $x, y \in F$. By Product formula 7.2 and 7.3 $\dim_H(F_r \times F_r) = 2\dim_H F = t$, so as $F \times F = \bigcup_{r=1}^{\infty}(F_r \times F_r)$, we have that $\dim_H(F \times F) = t$. Thus $F \times F$ is a subgroup of $\mathbb{R}^2$ with $\dim_H(F \times F) = t$.

Chapter 13

13.1 Let $f(x) = 2(1 - |2x - 1|)$, so if $x \le 1/2$, then $f(x) = 2(1 + 2x - 1) = 4x$, and if $x \ge 1/2$, then $f(x) = 2(1 - 2x + 1) = 4 - 4x$. Note that f has a maximum at $x = 1/2$ with $f(1/2) = 2$.

We note that there are two branches of f^{-1} defined on $[0, 1]$. It follows from the definition of f that these are the two functions $S_1, S_2 : [0, 1] \to \mathbb{R}$ defined by

$$S_1(x) = \frac{x}{4},\ S_2(x) = 1 - \frac{x}{4}.$$

These functions satisfy $f(S_1(x)) = f(S_2(x)) = x$, for $x \in [0, 1]$. Also, S_1 and S_2 are both contractions, since $|S_i(x) - S_i(y)| = |x - y|/4$, for $x, y \in [0, 1]$ and $i = 1, 2$, and so it follows from Theorem 9.1 that there is a compact set F satisfying $F = S_1(F) \cup S_2(F)$ given by $F = \bigcap_{k=0}^{\infty} S^k([0, 1])$. Clearly F is invariant for f since $f(F) = f(S_1(F)) \cup f(S_2(F)) = F \cup F = F$.

We now show that F is a repeller for f. We begin by noting that, if $x < 0$, then $f^n(x) \to -\infty$ as $n \to \infty$. Also, if $x > 1$, then $f(x) < 0$ and so $f^n(x) \to -\infty$ as $n \to \infty$. Thus any repeller for f must be contained in $[0, 1]$. If $x \in [0, 1] \backslash F$, then, for some positive integer k, $x \notin S^k([0, 1])$ and so $f^k(x) \notin [0, 1]$. Thus $f^n(x) \to -\infty$ as $n \to \infty$, for any $x \notin F$, and so F is indeed a repeller for f.

We now show that f is chaotic on F by denoting the points of F by $x_{i_1, i_2, \ldots} = \bigcap_{k=1}^{\infty} S_{i_1} \circ S_{i_2} \cdots S_{i_k}([0, 1])$ with $i_j = 1, 2$. We begin by noting that $|x_{i_1, i_2, \ldots} - x_{i'_1, i'_2, \ldots}| \le 4^{-k}$ if $(i_1, \ldots, i_k) = (i'_1, \ldots, i'_k)$ and that

$f(x_{i_1,i_2,\ldots}) = x_{i_2,i_3,\ldots}$. Now suppose that the sequence $(i_1, i_2, \ldots)$ is an infinite sequence with every finite sequence of 1s and 2s appearing as a consecutive block of terms. In this case, the orbit $\{f^k(x_{i_1,i_2,\ldots})\}$ is dense in F since, if $x_{i'_1,i'_2,\ldots} \in F$ and $q \in \mathbb{Z}^+$, then there exists $k \in \mathbb{Z}^+$ such that $(i'_1, \ldots, i'_q) = (i_{k+1}, \ldots, i_{k+q})$ and hence

$$|f^k(x_{i_1,i_2,\ldots}) - x_{i'_1,i'_2,\ldots}| = |x_{i_{k+1},i_{k+2},\ldots} - x_{i'_1,i'_2,\ldots}| \le 4^{-q}.$$

Now let $x_{i_1,i_2,\ldots}$ denote any point in F. The point $x_{i_1,\ldots,i_k,i_1,\ldots,i_k,i_1,\ldots}$ is a periodic point of f in F and

$$|x_{i_1,i_2,\ldots} - x_{i_1,\ldots,i_k,i_1,\ldots,i_k,i_1,\ldots}| \le 4^{-k}.$$

Thus the periodic points of f are dense in F.

Finally we show that the iterates of f have sensitive dependence on initial conditions. Again, let $x_{i_1,i_2,\ldots}$ denote any point in F and let $x_{i'_1,i'_2,\ldots}$ be another point in F with $(i_1, \ldots, i_k) = (i'_1, \ldots, i'_k)$ but $i_{k+1} \ne i'_{k+1}$. One of $f^k(x_{i_1,i_2,\ldots})$, $f^k(x_{i'_1,i'_2,\ldots})$ belongs to $[0, 1/4]$ whilst the other one belongs to $[3/4, 1]$. Thus

$$|f^k(x_{i_1,i_2,\ldots}) - f^k(x_{i'_1,i'_2,\ldots})| \ge 1/2$$

even though

$$|x_{i_1,i_2,\ldots} - x_{i'_1,i'_2,\ldots}| \le 4^{-k}.$$

F is the attractor of the similarities S_1 and S_2 given above, which have ratios $c_1 = c_2 = 1/4$ and satisfy the open set condition (9.11) with $V = (0, 1)$ (since the sets $S_1(V) = (0, 1/4)$ and $S_2(V) = (3/4, 1)$ are disjoint and contained in V). Thus it follows from Theorem 9.3 that $\dim_H F = \dim_B F = s$, where s is given by

$$1 = \sum_{i=1}^{2} c_i^s = 2(1/4)^s.$$

Thus $s \log(1/4) = \log(1/2)$ and so

$$s = \frac{\log(1/2)}{\log(1/4)} = \frac{\log(1/2)}{\log(1/2)^2} = \frac{1}{2}.$$

13.2 Inverting each of the three parts of the mapping defining f we get an associated IFS $\{S_1, S_2, S_3\}$ on $[0, 5]$ by taking

$$S_1(x) = \frac{1}{5}x, \quad S_2(x) = 2 - \frac{1}{5}x, \quad S_3(x) = 2 + \frac{1}{5}x.$$

Then $f(S_i(x)) = x$ for $i = 1, 2, 3$ and $x \in [0, 5]$.

The S_i are contracting similarities, so the IFS has an attractor F satisfying $F = S_1(F) \cup S_2(F) \cup S_3(F)$, with $F = \bigcap_{k=0}^{\infty} S^k([0,5])$. From the definition of the S_i as the branches of f^{-1}, we see that $f(F) = F$. To see that F is a repeller for f, note that if $x > 5$ then $f(x) = 5x - 10 = 3x + 2x - 10 > 3x$, so $f^k(x) \geq 3^k x \to \infty$, and if $x < 0$ then $f(x) = 5x$ so $f^k(x) \leq 5^k x \to -\infty$. If $x \in [0,5] \setminus F$ then $x \notin \bigcap_{k=0}^{\infty} S^k([0,5])$, so that $f^k(x) \notin [0,5]$ for some positive integer k, so either $f^k(x) \to \infty$ or $f^k(x) \to -\infty$. Thus all points outside F are iterated to $\pm\infty$, so F is a repeller.

The open set condition holds for the IFS $\{S_1, S_2, S_3\}$ taking $(0,5)$ as the open set, with $S_1(0,5) = (0,1)$, $S_2(0,5) = (1,2)$, $S_3(0,5) = (2,3)$; since each S_i is a similarity of ratio $1/5$, Theorem 9.3 gives that $\dim_{\mathrm{H}} F = s$ where $3 \times 5^{-s} = 1$, that is $s = \log 3/\log 5$.

13.3 This is similar to the argument for the logistic map for large λ. Assuming $\lambda > 1$, Write $a = \frac{1}{\pi}\sin^{-1}\frac{1}{\lambda}$ so that f_λ maps each of the intervals $[0,a]$ and $[1-a, 0]$ monotonically onto $[0,1]$. Inverting the restriction of f_λ to each of these intervals we get an associated IFS $\{S_1, S_2\}$ on $[0,1]$ given by

$$S_1(x) = \frac{1}{\pi}\sin^{-1}\frac{x}{\lambda}, \quad S_2(x) = 1 - \frac{1}{\pi}\sin^{-1}\frac{x}{\lambda}.$$

Then $f(S_i(x)) = x$ for $i = 1, 2$ and $x \in [0,1]$.

Differentiating, $|S_i'(x)| = \pi^{-1}(\lambda^2 - x^2)^{-1/2}$, for $i = 1, 2$, so using the mean value theorem.

$$\frac{1}{\pi\lambda} = \inf_{x\in[0,1]} |S_i'(x)| \leq \frac{|S_i(x) - S_i(y)|}{|x-y|} \leq \sup_{x\in[0,1]} |S_i'(x)| = \frac{1}{\pi\sqrt{\lambda^2 - 1}} \qquad (*)$$

for $x \neq y$. In particular, if $(1 + \pi^{-2})^{1/2} < \lambda$, S_1 and S_2 are contractions, so the IFS has an attractor F satisfying $F = S_1(F) \cup S_2(F)$ with $F = \bigcap_{k=0}^{\infty} S^k([0,1])$. From the definition of the S_i as the branches of the inverse of f_λ, we see that $f_\lambda(F) = F$. To see that F is a repeller for f_λ, note that if $x \in [0,1] \setminus F$ then $x \notin \bigcap_{k=0}^{\infty} S^k([0,1])$, so that $f_\lambda^k(x) \notin [0,1]$ for some positive integer k. Thus all points outside F are iterated to outside $[0,1]$, so F is a repeller.

Since $S_1([0,1])$ and $S_2([0,1])$ are disjoint, Propositions 9.6 and 9.7 give $s \leq \dim_{\mathrm{H}} F \leq t$, where

$$2(\pi\lambda)^{-s} = 1 = 2(\pi(\lambda^2 - 1)^{1/2})^{-t}$$

by $(*)$, giving

$$\log 2/\log(\pi\lambda) \leq \dim_{\mathrm{H}} F \leq \log 2/\log(\pi(\lambda^2 - 1)^{1/2}).$$

Thus when λ is large, $\dim_{\mathrm{H}} F \simeq \log 2/\log(\pi\lambda)$.

13.4 Suppose that $f_\lambda^n(x) \to l$ as $n \to \infty$. Then $f_\lambda^{n+1}(x) \to l$ as $n \to \infty$. Since

$$f_\lambda^{n+1}(x) = \lambda f_\lambda^n(x)(1 - f_\lambda^n(x)),$$

it follows that $l = \lambda l(1-l)$. Thus either $l = 0$ or $1 = \lambda(1-l)$; that is, either $l = 0$ or $l = 1 - 1/\lambda$.

If $x \in (0, 1)$, then

$$0 < f_{1/2}(x) = \frac{1}{2}x(1-x) < x/2.$$

Thus $f_{1/2}^n(x)$ is a decreasing sequence converging to 0.

Note that $f_2(x) = 2x(1-x)$. Thus, if $x \in (0, 1/2)$, then

$$x < f_2(x) < 1/2.$$

Thus $f_2^n(x)$ is an increasing sequence which is bounded above. It follows that $f_2^n(x)$ converges, so from the first part of the question, that $f_2^n(x)$ converges to $1 - 1/\lambda = 1/2$. If $x \in (0, 1/2)$, then $0 < f_2(x) < x$ and $f_\lambda^n(x)$ increases to $1/2 = 1 - 1/\lambda$. If $x \in (1/2, 1)$, then $f_2(x) \in (0, 1/2)$ and so $f_2^n(x)$ also converges to $1/2$. Finally, $f_2(1/2) = 1/2$ and so $f_2^n(1/2)$ trivially converges to $1/2$.

Finally, we consider f_4. We note that 0 is an unstable fixed point of f_4 (since $f_4(0) = 0$ and $f_4'(0) = 4 > 1$) and so, if $f_4^n(x)$ converges to 0, then there must be some integer m for which $f_4^m(x) = 0$. Now $f_4(1) = 0$ and $f_4(1/2) = 1$ so that $f_4^2(1/2) = 0$. There are no other non-zero preimages of 0 and 1 under f_4 and hence 1/2 is the only non-zero preimage of 0 under f_4^2. If $0 < x < 1$, then there are exactly two points in $(0, 1)$ which map to x under f_4. Thus, for each positive integer k, there are exactly 2^k points in $(0, 1)$ which map to x under f_4^k and hence to 0 under f_4^{k+2}. These are the only points which converge to 0 under iteration.

Similarly, $3/4 = 1 - 1/\lambda$ is an unstable fixed point of f_4 and so, if $f_4^n(x)$ converges to $3/4$, then there must be some integer m for which $f_4^m(x) = 3/4$. For each k, there are exactly 2^k points in $(0, 1)$ which map to $3/4$ under f_4^k and hence converge to $3/4$ under iteration.

We have shown that there are countably many points $x \in (0, 1)$ for which $f_4^n(x) \to 0$ as $n \to \infty$ and countably many points $x \in (0, 1)$ for which $f_4^n(x) \to 3/4$ as $n \to \infty$. This leaves infinitely many points $x \in (0, 1)$ for which $f_4^n(x)$ cannot converge (since any convergent sequence of iterates must converge to either 0 and 3/4).

13.5 (i) We use proof by induction. If $k = 0$, then $x_k = \frac{1}{2}(1 - \exp(2^k a)) = \frac{1}{2}(1 - \exp a) = x$. Thus the formula is correct for $k = 0$. If the formula is correct for k, then

$$\begin{aligned} x_{k+1} &= f_2(x_k) = 2x_k(1-x_k) \\ &= 2(1/2)(1-\exp(2^k a))(1-(1-\exp(2^k a))/2) \\ &= (1-\exp(2^k a))(1/2+\exp(2^k a)/2) \\ &= (1/2)(1-\exp(2^k a))(1+\exp(2^k a)) \\ &= (1/2)(1-\exp(2^{k+1} a)). \end{aligned}$$

Thus, if the formula is correct for k, then it is also correct for $k+1$. Since we have shown that the formula is correct for $k=0$, it follows by induction that the formula is correct for all k.

(ii) We will use proof by induction. If $k=0$, then $x_k = \sin^2(\pi a) = x$. Thus the formula is correct for $k=0$. If the formula is correct for k, then

$$\begin{aligned} x_{k+1} &= f_4(x_k) = 4x_k(1-x_k) \\ &= 4\sin^2(2^k\pi a)(1-\sin^2(2^k\pi a)) \\ &= 4\sin^2(2^k\pi a)\cos^2(2^k\pi a) \\ &= [2\sin(2^k\pi a)\cos(2^k\pi a)]^2 \\ &= \sin^2(2^{k+1}\pi a). \end{aligned}$$

Thus, if the formula is correct for k, then it is also correct for $k+1$. Since we have shown that the formula is correct for $k=0$, it follows by induction that the formula is correct for all k.

If $a = 0\cdot a_1a_2\ldots$ in binary form, then

$$\begin{aligned} x_k &= \sin^2(2^k\pi a) = \sin^2(a_1a_2\ldots a_k\cdot a_{k+1}\ldots\pi) \\ &= \sin^2(0\cdot a_{k+1}\ldots\pi) \end{aligned}$$

using the periodicity of the sine function. So, if $a = 0\cdot a_1\ldots a_pa_1\ldots a_pa_1\ldots$, then

$$x_p = \sin^2(0\cdot a_1\ldots a_pa_1\ldots a_pa_1\ldots\pi) = x;$$

that is, $f_4^p(x) = x$.

We must now show that it is possible to choose $a_1,\ldots,a_p$ in such a way as to ensure that the periodic point x is unstable; that is, $|(f_4^p)'(x)| > 1$. Now

$$(f_4^p)'(x) = \prod_{i=0}^{p-1} f_4'(f_4^i(x)),$$

by the chain rule, and, for any $\hat{x}$, we have $f_4(\hat{x}) = 4\hat{x}(1-\hat{x})$ so that $f_4'(\hat{x}) = 4 - 8\hat{x}$. If $\hat{x} = \sin^2(\pi\hat{a}')$, then

$$f_4'(\hat{x}) = 4(1 - 2\sin^2(\pi\hat{a})) = 4\cos(2\pi\hat{a}).$$

Thus

$$(f_4^p)'(x) = 4^p \prod_{i=1}^{p} \cos(0 \cdot a_i \ldots a_p a_1 \ldots a_p a_1 \ldots \pi).$$

If p is odd, then we put

$$a_1 \ldots a_p = 1010\ldots 101$$

and, if p is even, then we put

$$a_1 \ldots a_p = 11010\ldots 101.$$

If

$$0 \cdot a_i \ldots a_p a_1 \ldots a_p a_1 \ldots \pi > 0 \cdot 101\pi = 5\pi/8$$

or

$$0 \cdot a_i \ldots a_p a_1 \ldots a_p a_1 \ldots \pi < 0 \cdot 011\pi = 3\pi/8,$$

then

$$|\cos(0 \cdot a_i \ldots a_p a_1 \ldots a_p a_1 \ldots \pi)| \geq |\cos(3\pi/8)| = |\cos(5\pi/8)| > 0.38.$$

Otherwise, $0 \cdot a_i \ldots a_p a_1 \ldots a_p a_1 \ldots \pi$ is close to $\pi/2$ and

$$\begin{aligned}
&|\cos(0 \cdot a_i \ldots a_p a_1 \ldots a_p a_1 \ldots \pi)\cos(0 \cdot a_{i+1} \ldots a_p a_1 \ldots a_p a_1 \ldots \pi)| \\
&\geq |\cos(0 \cdot 011101\ldots\pi)\cos(0 \cdot 11101\pi)| \\
&\geq |\cos(0 \cdot 01111\pi)\cos(0 \cdot 111\pi)| \\
&= |\cos(15\pi/32)\cos(7\pi/8)| > 0.098 \times 0.92 > 0.09.
\end{aligned}$$

Now $(0 \cdot 38)^2 > 0 \cdot 09$ and hence

$$|(f_4^p)'(x)| > (0.09)^{p/2} \times 4^p = (1 \cdot 2)^p > 1$$

as required.

Finally, we show that f_4 has a dense orbit. Suppose that $(a_1, a_2, \ldots)$ is an infinite sequence with every finite sequence of 0s and 1s appearing as a consecutive block of terms. We claim that the orbit $\{f_4^k(x)\}$ is dense in $[0, 1]$, if $x = \sin^2(\pi a)$, where $a = 0 \cdot a_1 a_2 \ldots$. To show that this is true, we take another point $x' \in [0, 1]$ with $x' = \sin^2(\pi a')$, where $a' =$

$0 \cdot a_1' a_2' \ldots$ in binary form. For each positive integer q, there exists k such that $(a_1', \ldots, a_q') = (a_{k+1}, \ldots, a_{k+q})$. Thus

$$\begin{aligned} |f_4^k(x) - x'| &= |\sin^2(0 \cdot a_{k+1}a_{k+2}\ldots\pi) - \sin^2(0 \cdot a_1' a_2' \ldots \pi)| \\ &= |\sin^2((a' + \epsilon)\pi) - \sin^2(a'\pi)|, \end{aligned}$$

where $|\epsilon| \le 2^{-q}$. Since we can choose q to be arbitrarily large, it follows that the orbit $\{f_4^k(x)\}$ comes arbitrarily close to the point x'.

13.6 Let $f : E \to E$ be given by

$$f(x, y) = \begin{cases} (2x, \lambda y) & \left(0 \le x \le \dfrac{1}{2}\right) \\ (2x - 1, \mu y + 1/2) & \left(\dfrac{1}{2} < x \le 1\right) \end{cases}$$

where $0 < \lambda, \mu < 1/2$, and let $E_k = f^k(E)$. Then E_k is a decreasing sequence of sets and $F = \bigcap_{k=0}^{\infty} E_k$ satisfies $f(F) = F$. Each set E_k is made up of 2^k horizontal strips of height at most M^k, where $M = \max(\lambda, \mu)$. Thus F is made up of horizontal lines with at least one line in each strip of E_k. If $(x, y) \in E$, then $f^k(x, y) \in E_k$ and so the distance of $f^k(x, y)$ from F is at most M^k. Since $M < 1/2$, it follows that every point in E is attracted to F.

We now find the Hausdorff dimension of F. We begin by noting that $(0, 1] \times F_1 \subset F \subset [0, 1] \times F_1$, where F_1 is the attractor of the mappings $S_1, S_2 : [0, 1] \to \mathbb{R}$ defined by

$$S_1(y) = \lambda y, \ S_2(y) = \mu y + 1/2.$$

Now S_1 and S_2 are contractions with ratios λ and μ. Since they also satisfy the open set condition (9.11) with $V = (0, 1)$, it follows from Theorem 9.3 that $\dim_H F_1 = \dim_B F_1 = s$, where s is given by $1 = \lambda^s + \mu^s$. It follows from Corollary 7.4 that

$$\dim_H((0, 1] \times F_1) = \dim_H([0, 1] \times F_1) = 1 + s$$

and hence, by monotonicity, $\dim_H F = 1 + s$, where $\lambda^s + \mu^s = 1$.

13.7 It may be verified computationally that the four sides of the quadrilateral specified are mapped onto parabolae which lie inside the quadrilateral.

Iterates of a typical point give a plot similar to Figure 13.8.

13.8 This is very similar to the argument for the solenoid in Section 13.4. With

$$f(\phi, w) = (3\phi(\text{mod}2\pi), aw + \tfrac{1}{2}\hat{\phi}),$$

we see that $f^k(D)$ is a solid tube of radius a^k going round D 3^k times. The set $F = \bigcap_{k=1}^{\infty} f^k(D)$ is compact and invariant under f and attracts all points of D.

To find the dimensions of F, let P_ϕ be the half-plane bounded by the central axis L and cutting the central circle of D at $(\phi, 0)$. Then $f^k(C)$ is a smooth closed curve traversing the torus 3^k times with total length at most $3^k c$, where c is independent of k. The set $f^k(D)$ is a fattening of the curve $f^k(C)$ to a tube of radius a^k, so it may be covered by a collection of balls of radii $2a^k$ spaced at intervals a^k along $f^k(C)$. Then $2 \times 3^k c a^{-k}$ such balls will suffice, so applying Proposition 4.1 in the usual way, we get $\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F \leq 1 - \log 3/\log a$.

For a lower estimate, we consider the sections $F \cap P_\phi$ for each ϕ. The set $f(D) \cap P_\phi$ contains three discs of radius a situated symmetrically with centres at least $\frac{1}{4}$ apart. Each of these discs contains three discs of $f^2(D) \cap P_\phi$ of radius a^2 with centres at least $\frac{1}{4}a$ apart, and so on. Thus we may regard $F \cap P_\phi$ as formed by a standard nested construction, the kth stage consisting of 3^k discs of radius a^k with centres separated by at least $\frac{1}{4}a^{k-1}$. We may define a mass distribution μ on $F \cap P_\phi$ such that each of the 3^k level-k discs has mass 3^{-k}. A standard application of the mass distribution principle gives that $\dim_{\mathrm{H}}(F \cap P_\phi) \geq -\log 3/\log a$. Since F is built up from sections $F \cap P_\phi$ $(0 \leq \phi < 2\pi)$, a higher dimensional version of Proposition 7.9 gives that $\dim_{\mathrm{H}} F \geq 1 - \log 3/\log a$, so that $\dim_{\mathrm{H}} F = \overline{\dim}_{\mathrm{B}} F = 1 - \log 3/\log a$.

The chaotic behaviour of f may be examined by noting that if $\phi/2\pi = 0.a_1a_2\ldots$ to base 3, then $f^k(\phi, w) = (\phi_k, v_k)$, where $\phi_k/2\pi = 0.a_{k+1}a_{k+2}\ldots$ and where the integer with base 3 representation $a_k a_{k-1}\ldots a_{k-d+1}$ determines which of the 3^d discs of $f^d(D) \cap P_{\phi_k}$ the point v_k belongs to for $d \leq k$. By choosing digits $a_1, a_2, \ldots$ suitably, it is easy to produce orbits that are dense in f or which are periodic.

13.9 Suppose that $x = f(t) + \epsilon$ for some $\epsilon \neq 0$. Then

$$\begin{aligned}
h(t,x) &= (\lambda t, \lambda^{2-s}(x - g(t))) \\
&= (\lambda t, \lambda^{2-s}(\epsilon + f(t) - g(t))) \\
&= \left(\lambda t, \lambda^{2-s}\left(\epsilon + \sum_{k=1}^{\infty} \lambda^{(s-2)k} g(\lambda^k t)\right)\right) \\
&= \left(\lambda t, \lambda^{2-s}\epsilon + \sum_{k=1}^{\infty} \lambda^{(s-2)(k-1)} g(\lambda^k t)\right)
\end{aligned}$$

$$= \left(\lambda t, \lambda^{2-s}\epsilon + \sum_{k=0}^{\infty} \lambda^{(s-2)k} g(\lambda^{k+1}t)\right)$$
$$= (\lambda t, \lambda^{2-s}\epsilon + f(\lambda t)).$$

So, if (t, x) is at a vertical distance ϵ from graph$f$, then $h(t, x)$ is at a vertical distance $\lambda^{2-s}\epsilon > \epsilon$ from graph$f$. It follows that the distance of $h^n(t, x)$ from graph$f$ tends to infinity as $n \to \infty$ and so graph$f$ is indeed a repeller for h as claimed.

13.10 With the notation of Section 10.1, take μ to be the probability measure P on [0,1] defined in (10.2). With $F = F(p_0, \ldots, p_{m-1})$, Proposition 10.1 showed that $\mathsf{P}(F) = 1$ and that $\dim_{\mathrm{H}} F = -(\sum_{i=0}^{m-1} p_i \log p_i)/\log m$ which is strictly < 1 provided that the p_i are not all equal, giving by (13.16) $\dim_{\mathrm{H}}\mathsf{P} \leq \dim_{\mathrm{H}} F < 1$. However, F is dense in $[0, 1]$, so that $\dim_{\mathrm{H}}\mathrm{spt}\mathsf{P} = 1$, giving $\dim_{\mathrm{H}}\mathsf{P} < \dim_{\mathrm{H}}\mathrm{spt}\mathsf{P}$, as required.

13.11 Differentiating f we see that the Jacobian matrix of f is

$$J \equiv \frac{\partial f}{\partial(x, y)} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

at all (x, y) such that $f(x, y)$ is not on the boundary of the unit square E. Since the determinant of J is 1, the map is area preserving.

The eigenvalues of J are $(3 \pm \sqrt{5})/2$, so we may choose orthogonal axes such that, with respect to these axes, J is represented by the diagonal matrix $\begin{pmatrix} (3+\sqrt{5})/2 & 0 \\ 0 & (3-\sqrt{5})/2 \end{pmatrix}$. By the chain rule, the derivative of the kth iterate f^k is J^k, which with respect to these axes is

$$\begin{pmatrix} ((3+\sqrt{5})/2)^k & 0 \\ 0 & ((3-\sqrt{5})/2)^k \end{pmatrix}$$

at all (x, y) such that $f^j(x, y)$ is not on the boundary of the unit square E for all $j = 1, 2, \ldots, k$, which is the case for $\mathcal{L}^2$-almost all $(x, y) \in E$. It follows from (13.9) that, for almost all (x, y), the Liapunov exponents are

$$\log(3 + \sqrt{5})/2 \quad \text{and} \quad \log(3 - \sqrt{5})/2,$$

and thus these values are the Liapounov exponents of the cat map.

Chapter 14

14.1 We may choose distinct points $x_1, x_2 \in E$ and a number $0 < r_1 \leq 2^{-1}$ such that the discs $B(x_1, r_1)$ and $B(x_2, r_1)$ are disjoint. Since x_1 and x_2 are not isolated, we may, for $i = 1, 2$, choose $x_{i,1}, x_{i,2} \in E \cap B(x_i, r_1)$ and a number $0 < r_2 \leq 2^{-2}$ such that the discs $B(x_{i,1}, r_2)$ and $B(x_{i,2}, r_2)$ are contained in $B(x_i, r_1)$ and are disjoint. Proceeding in this way, we may find distinct points $x_{i_1,i_2,\dots,i_k} \in E$ and discs $B(x_{i_1,i_2,\dots,i_k}, r_k)$ with $r_k \leq 2^{-k}$ such that $B(x_{i_1,i_2,\dots,i_k,1}, r_{k+1})$ and $B(x_{i_1,i_2,\dots,i_k,2}, r_{k+1})$ are disjoint subdiscs of $B(x_{i_1,i_2,\dots,i_k}, r_k)$.

For every infinite sequence $i_1, i_2, \dots$ of 1s and 2s, let $x_{i_1,i_2,\dots} = \bigcap_{i=0}^{\infty} B(x_{i_1,i_2,\dots,i_k}, r_k)$, which is a single point as the intersection of closed discs of radii tending to 0. Since E is closed, $x_{i_1,i_2,\dots} \in E$ as the limit as $k \to \infty$ of $x_{i_1,i_2,\dots,i_k} \in E$. Moreover, the $x_{i_1,i_2,\dots}$ are distinct, for if $i_1, i_2, \dots, i_{k-1}, i_k, i_{k+1}, \dots$ and $i_1, i_2, \dots, i_{k-1}, j_k, j_{k+1}, \dots$ are distinct sequences of 1s and 2s with $i_k \neq j_k$, then $x_{i_1,i_2,\dots,i_{k-1},i_k,i_{k+1},\dots} \in B(x_{i_1,i_2,\dots,i_k}, r_k)$ and $x_{i_1,i_2,\dots,i_{k-1},j_k,j_{k+1},\dots} \in B(x_{i_1,i_2,\dots,i_{k-1},j_k}, r_k)$ are in distinct level k discs. There are uncountably many sequences $i_1, i_2, \dots$ (such sequences are in correspondence with real numbers in [0, 1] numbers expressed to base 2), so there are uncountably many points $x_{i_1,i_2,\dots} \in E$.

14.2 We use the fact that, if $f(z) = (\alpha^2 z^2 + 2\alpha\beta z + \beta^2 + c - \beta)/\alpha$, then $J(f) = h^{-1}(J(f_c))$, where $f_c(z) = z^2 + c$ and $h(z) = \alpha z + \beta$, see Section 14.2. Here $f(z) = z^2 + 4z + 2$ and so $\alpha^2/\alpha = 1$, $2\alpha\beta/\alpha = 4$ and $(\beta^2 + c - \beta)/\alpha = 2$. Thus $\alpha = 1$, $\beta = 2$ and $c = 2 + \beta - \beta^2 = 0$. So $f_c(z) = z^2$ and hence $J(f_c)$ is the unit circle. Since $h^{-1}(z) = (z - \beta)/\alpha = z - 2$, it follows that $J(f)$ is the circle of radius 1 whose centre is at -2.

14.3 We recall from Section 14.2 that if $h(z) = \alpha z + \beta$ and $f(z) = h^{-1}(f_c(h(z)))$ then $J(f) = h^{-1}(J(f_c))$. Taking $h(z) = z + i$ so $h^{-1}(z) = z - i$, this gives that the Julia set of $f(z) = (z + i)^2 + c - i = z^2 + 2iz - 1 + c - i$ is $J(f_c) - i$, which is congruent to $J(f_c)$. This Julia set is connected if and only if $c \in M$, thus the Julia set of $f(z) = z^2 + 2iz + b$ is connected if and only if $c \in M$ where $b = -1 + c - i$, that is if and only if $b + 1 + i \in M$.

14.4 If $|c| \leq \frac{1}{4}$ and $|z| \leq \frac{1}{2}$, then

$$|f_c(z)| = |z^2 + c| \leq |z|^2 + |c| \leq \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}.$$

Thus if $|z| \leq \frac{1}{2}$ then, applying this inductively, $|f_c^k(z)| \leq \frac{1}{2}$ for all $k \in \mathbb{Z}^+$. In particular, $|f_c^k(z)| \not\to \infty$, so $z \in K(f_c)$, the filled in Julia set. Thus $B(0, \frac{1}{2}) \subset K(f_c)$.

On the other hand, if $|c| \le \frac{1}{4}$ and $|z| \ge 2$, then

$$|f_c(z)| = |z^2 + c| \ge |z|^2 - |c| \ge 2|z| - |c| \ge 2|z| - \frac{1}{2}|z| = \frac{3}{2}|z|,$$

since $|z| \ge 2 \ge 2|c|$. Applying this repeatedly, if $|z| \ge 2$ then $|f_c^k(z)| \ge \frac{3}{2}|f_c^{k-1}(z)| \ge \ldots \ge (\frac{3}{2})^k|z| \ge 2(\frac{3}{2})^k \to \infty$. We conclude that if $|z| \ge 2$ then $z \notin K(f_c)$, so $K(f_c) \subset B(0, 2)$.

From $B(0, \frac{1}{2}) \subset K(f_c) \subset B(0, 2)$ we conclude that the Julia set, which is the boundary of $K(f_c)$, lies in the annulus $B(0, 2)\backslash B^o(0, \frac{1}{2})$.

14.5 The fixed points of f are given by $f(z) = z^2 - 2 = z$, so are $z = -1, 2$. Since $f'(z) = 2z$ we have that $|f'(2)| = 4 > 1$, so 2 is a repelling fixed point.

By Theorem 14.10, $2 \in J(f)$, so by Corollary 14.8(b), $J(f)$ is the closure of $\bigcup_{k=1}^{\infty} f^{-k}(2)$. If $w \in [-2, 2]$ then $f^{-1}(w) = (w + 2)^{1/2} \in [-2, 2]$, so $f^{-k}(w) \in [-2, 2]$, and in particular $f^{-k}(2) \in [-2, 2]$, for all $k \in \mathbb{Z}^+$. Thus $J(f)$ is contained in the closure of $[-2, 2]$, that is $J(f) \subset [-2, 2]$.

Now observe that $f(0) = -2$ and $f^2(0) = f(-2) = 2$. Since 2 is a fixed point of f, it follows that $f^k(0) = 2$ for $k = 2, 3, \ldots$, so $f^k(0) \not\to \infty$. Thus $-2 \in M$ by Theorem 14.14, so the Julia set $J(f)$ is connected. But $2 \in J(f)$ and $-2 \in J(f)$ (since $f^{-1}(2) = \{2, -2\}$), and the Julia set $J(f)$ is a connected subset of the interval $[-2, 2]$ containing its endpoints -2 and 2, so this requires $J(f) = [-2, 2]$.

14.6 For $z \in \mathbb{C}$ we have $f_c(-z) = f_c(z)$, so $f_c^k(-z) = f_c^k(z)$ for all $k \in \mathbb{Z}^+$. Thus $f_c^k(-z) \to \infty$ if and only if $f_c^k(z) \to \infty$, so that $-z \in K(f_c)$ if and only if $z \in K(f_c)$. Thus the filled-in Julia set $K(f_c)$ is symmetric about the origin, and its boundary, the Julia set $J(f_c)$, is also symmetric about the origin.

14.7 Let c be real with $c > \frac{1}{4}$. Then $f_c^k(z)$ is real for all $k \in \mathbb{Z}^+$ and real z. In particular, for real z,

$$f_c(z) - z = z^2 + c - z = \left(z - \frac{1}{2}\right)^2 + \left(c - \frac{1}{4}\right) \ge \left(c - \frac{1}{4}\right).$$

Applying this repeatedly,

$$f_c^k(z) \ge z + k\left(c - \frac{1}{4}\right) \to \infty.$$

We conclude that $z \notin J(f_c)$ if z is real.

The Julia set $J(f_c)$ is non-empty (by Proposition 14.2) and symmetric about the origin (by Exercise 14.6), so there exists $z \in \mathbb{C}$ with $z, -z \in J(f_c)$. Since $J(f_c)$ contains no points on the real axis and z and $-z$ lie on opposite sides of the real axis, $J(f_c)$ is not connected. Thus by (14.4) $c \notin M$.

14.8 The fixed points of $f = f_c$ are given by $z^2 + c = z$, that is $z^2 - z + c = 0$ or $z = \frac{1}{2}(1 \pm (1 - 4c)^{1/2})$. The sum of these two distinct roots is 1, so we may choose one of the fixed points $w = \frac{1}{2}(1 + (1 - 4c)^{1/2})$, say, with $|w| > \frac{1}{2}$. Thus $|f'(w)| = 2|w| > 1$, so w is a repelling fixed point. The number w is real if and only if $(1 - 4c)^{1/2}$ is real, which is not the case if c is non-real. Hence $f'(w) = 2w$ is not real.

We know from Theorem 14.16 that if $|c| < \frac{1}{4}$ then the Julia set $J = J(f)$ is a simple closed curve. Suppose that this curve J has a tangent at w. Since $f(w) = w$ and $f(J) = J$, and f is analytic near w, f maps a neighbourhood of J containing w to a neighbourhood of J containing w. To a first order approximation (considering the Taylor series expansion of f around w), we have

$$f(w + z) = f(w) + f'(w)z + O(z^2) = w + f'(w)z + O(z^2).$$

In particular, f maps the tangent to J at w, which may be written parametrically as $\{w + tz_0 : t \in \mathbb{R}\}$ near w, onto a smooth curve that is tangential to J at w of the form $\{w + tf'(w)z_0 + O(t^2) : t \in \mathbb{R}\}$. This is only possible if this curve is tangential to the original tangent at w, that is if $f'(w)$ is real.

We conclude from the first part that if c is a non-real number with $|c| < \frac{1}{4}$ then $f'(w)$ is not real, so J does not have a tangent at w.

If J contains an arc A, then we may find $n \in \mathbb{Z}^+$ such that $f^n(w)$ is a interior point of the arc A, by Corollary 14.8. But f^n is a smooth locally bijective mapping that maps a non-differentiable arc of J containing w into A, so that A cannot be differentiable.

14.9 If $|c| \le \frac{1}{4}$ and $|z| \le \frac{1}{2}$, then using the triangle inequality

$$|f_c(z)| = |z^2 + c| \le |z|^2 + |c| \le \left(\frac{1}{2}\right)^2 + \frac{1}{4} = \frac{1}{2}.$$

If $|c| \le \frac{1}{4}$, applying this inductively gives $|f_c^k(0)| \le \frac{1}{2}$ for all $k \in \mathbb{Z}^+$. Thus $f_c^k(0) \not\to \infty$, so $c \in M$. Thus $B(0, \frac{1}{4}) \subset M$.

14.10 If $|c+1| \le \frac{1}{20}$, we note that $|c| \le \frac{21}{20}$. Thus if $|z| \le \frac{1}{10}$, then using the triangle inequality

$$\begin{aligned}
|f_c(f_c(z))| = |(z^2+c)^2+c| &= |z^4+2cz^2+c(c+1)| \\
&\le |z|^4 + 2|c||z|^2 + |c||c+1| \\
&\le \left(\frac{1}{10}\right)^4 + 2\frac{21}{20}\left(\frac{1}{10}\right)^2 + \frac{21}{20}\frac{1}{20} < \frac{1}{10}.
\end{aligned}$$

Thus if $|c+1| \le \frac{1}{20}$ then, applying this inductively, $|f_c^{2k}(0)| \le \frac{1}{10}$ for all $k \in \mathbb{Z}^+$. Thus $f_c^k(0) \not\to \infty$, so $c \in M$. Thus $B(-1, \frac{1}{20}) \subset M$.

14.11 We have

$$\begin{aligned}
|f_c(z)| = |z^2+c| \ge |z|^2 - |c| = |z|\left(|z| - \frac{|c|}{|z|}\right) &\ge |z|(2+\epsilon-1) \\
\ge |z|(1+\epsilon),
\end{aligned}$$

provided that $|z| \ge \max\{2+\epsilon, |c|\}$.

If $|c| > 2$ we may choose $\epsilon > 0$ such that $|c| > 2+\epsilon$, so noting that $f_c(0) = c$ and applying the above estimate inductively,

$$|f_c^k(0)| \ge (1+\epsilon)^k |c| \to \infty.$$

Thus $c \notin M$ by Theorem 14.14.

14.12 If c is such that $|f_c^k(0)| > 2$ for some k, then either

(i) $|c| > 2$ so $c \notin M$ by Exercise 14.11, or

(ii) $|c| \le 2$, and we may choose $\epsilon > 0$ such that $|f_c^k(0)| > 2+\epsilon > |c|$. Then applying the first part of Exercise 14.11 repeatedly to $z = f_c^k(0)$ and its iterates under f_c, gives $|f_c^{k+n}(0)| = |f_c^n(f_c^k(0))| \ge (1+\epsilon)^n |f_c^k(0)| \to \infty$, so $c \notin M$.

On the other hand, if $c \notin M$ then $|f_c^k(0)| \to \infty$, so $|f_c^k(0)| > 2$ for some k.

14.13 Let $f(z) = z^3 + cz$, so $f'(z) = 3z^2 + c$. The fixed points of f are $0, \pm\sqrt{1-c}$, with $f'(0) = 0$, $f'(\pm\sqrt{1-c}) = 3-2c$. Hence, provided $|c| < 1$, 0 is an attractive fixed point of f. But the Julia set of a polynomial f is a closed curve precisely when f has an attractive fixed point, see note before Theorem 14.16 (a proof along the lines of Theorem 14.16 works when this is the case).

14.14 The solution is similar to the proof of Theorem 14.15. Let $|c| > \sqrt{2}$. Let C be the circle $|z| = |c|$ and let D be its interior $|z| < |c|$. If we define $S_1, S_2, S_3 : D \to D$ to be the branches of $f^{-1}(z) = (z - c)^{1/3}$, then $S_1(D)$, $S_2(D)$ and $S_3(D)$ are the interiors of the three loops of the curve $f^{-1}(C)$.

We now let V be the disc $\{z : |z| < |2c|^{1/3}\}$ so that V just contains $f^{-1}(D)$. Note that, if $|c| > \sqrt{2}$, then $\overline{V} \subset D$ and so the sets $S_1(V), S_2(V), S_3(V)$ are all in V and the sets $S_1(\overline{V})$, $S_2(\overline{V})$, $S_3(\overline{V})$ are disjoint. For $i = 1, 2, 3$, we have

$$|S_i'(z)| = \frac{1}{3}|z - c|^{-2/3}$$

and so, if $z \in \overline{V}$,

$$\frac{1}{3}(|c| + |2c|^{1/3})^{-2/3} \leq |S_i'(z)| \leq \frac{1}{3}(|c| - |2c|^{1/3})^{-2/3}.$$

It now follows from a complex mean-value theorem that

$$\frac{1}{3}\left(|c| + |2c|^{1/3}\right)^{-2/3} \leq \frac{|S_i(z_1) - S_i(z_2)|}{|z_1 - z_2|} \leq \frac{1}{3}(|c| - |2c|^{1/3})^{-2/3},$$

for $i = 1, 2, 3$ and $z_1, z_2 \in \overline{V}$.

If $|c| > 2$ (for example), then the upper bound is less than 1 and so S_1, S_2 and S_3 are contractions on the disc $\overline{V}$. It follows from Theorem 9.1 that there is a unique attractor $F \subset \overline{V}$ satisfying

$$S_1(F) \cup S_2(F) \cup S_3(F) = F.$$

It follows from Propositions 9.6 and 9.7 that lower and upper bounds for $\dim_{\mathrm{H}} F$ are provided by the solutions of

$$3\left(\frac{1}{3}(|c| \pm |2c|^{1/3})^{-2/3}\right)^s = 1,$$

that is, by

$$s = \frac{\log 3}{\log 3 + (2/3)\log(|c| \pm |2c|^{1/3})}.$$

So, when $|c|$ is large,

$$\dim_{\mathrm{H}} F \sim \frac{3\log 3}{2\log|c|}.$$

It remains to show that $F = J(f)$. If $|c| > \sqrt{2}$ and $|z| \notin \overline{V}$, i.e. $|z| > |2c|^{1/3}$ then

$$|f(z)| = |z^3 + c| \geq |z|^3 - |c| \geq |z|^3 - \frac{1}{2}|z|^3 = \frac{1}{2}|z|^3 = |z|\frac{1}{2}|z|^2$$
$$\geq |z|\frac{1}{2}|2c|^{2/3} = \lambda|z|$$

where $\lambda = \frac{1}{2}|2c|^{2/3} > \frac{1}{2}|2|^{2/3}|2|^{1/3} = 1$. Iterating, it follows that $|f^k(z)| \geq \lambda^k|z| \to \infty$, so $z \notin J(f)$. Thus $J(f) \subset K(f) \subset \overline{V}$. It follows from Propositions 14.2 and 14.3 that $J(f)$ is the non-empty compact subset of $\overline{V}$ satisfying $J(f) = f^{-1}(J(f))$, that is, $J(f) = S_1(J(f)) \cup S_2(J(f)) \cup S_3(J(f))$ and so $J(f) = F$ as claimed.

14.15 If $f_c(z) = z^2 + c$, then the fixed points of f_c are the solutions of $z^2 + c = z$. Since $f_c'(z) = 2z$, we see that $z = re^{i\theta}$ is an attractive fixed point of f_c if and only if $0 \leq r < 1/2$, $0 \leq \theta < 2\pi$ and

$$c = z - z^2 = z(1-z) = re^{i\theta}(1 - re^{i\theta}).$$

Thus f_c has an attractive fixed point precisely when c lies inside the main cardioid of the Mandelbrot set.

14.16 Since $f_c(z) = z^2 + c$ and $f_c^2(z) = f_c(f_c(z))$, we have

$$\begin{aligned} f_c^2(z) - z &= (z^2 + c)^2 + c - z \\ &= z^4 + 2cz^2 - z + c^2 + c \\ &= (z^2 - z + c)(z^2 + z + c + 1). \end{aligned}$$

Now z is a periodic point of f_c of period 2 if and only if $f_c^2(z) = z$ and $f_c(z) \neq z$. Thus we are looking for the solutions of $f_c^2(z) - z = 0$ which are not solutions of $f_c(z) - z = z^2 - z + c = 0$. In other words, we are interested in the solutions of $z^2 + z + c + 1 = 0$. By the chain rule

$$(f_c^2)'(z) = f_c'(f_c(z))f_c'(z) = 2(z^2 + c)2z$$

and so z is an attractive fixed point of f_c^2 if and only if

$$|(z^2 + c)z| < 1/4 \text{ and } z^2 + z + c + 1 = 0.$$

Using the second of these conditions to rewrite the first, we find that f_c^2 has an attractive fixed point if and only if

$$|(z^2 + c)z| = |(z+1)z| = |z^2 + z| = |c + 1| < 1/4.$$

Thus f_c^2 has an attractive fixed point precisely when c belongs to the main bud of the Mandelbrot set; this is the region labelled 2 in Figure 14.8.

14.17 Assume, for a contradiction, that c is not in the basin of attraction $A(w)$ of the (finite) attractive fixed point w of f_c. Let U be an open disc with $w \in U \subset A(w)$. Then $f_c^k(c) \notin U$ for all $k = 0, 1, 2, \ldots$. Thus for each k we may select a branch of the inverse f_c^{-k} on U to be a continuous analytic function with $f_c^{-k}(w) = w$. If $z \in f_c^{-k}(U)$ then $f_c^k(z) \in U \subset A(w)$, so $z \in A(w)$; thus $f_c^{-k}(U) \subset A(w)$ for all k. Since $A(w)$ is a bounded subset of $\mathbb{C}$, Montel's theorem implies that $\{f_c^{-k}\}_{k=0}^{\infty}$ is a normal family on U. However, since w is a repelling fixed point of f_c^{-1}, no subsequence of $f_c^{-k}(z)$ can be uniformly convergent to an analytic function near w (since $(f_c^{-k})'(w) = ((f_c^{-1})'(w))^k \to \infty$ by the chain rule), so $\{f_c^{-k}\}_{k=0}^{\infty}$ cannot be normal by the definition of a normal family. We conclude that c must be in the basin of attraction $A(w)$ of w.

Since c cannot be in the basin of attraction of more than one point, it follows that f_c has at most one (finite) attractive fixed point.

Now let f be any polynomial on $\mathbb{C}$ and let $A(w)$ be the basin of attraction of some (finite) attractive fixed point w of f. Assume, for a contradiction, that $c \notin A(w)$ for all $c \in \mathbb{C}$ such that $f'(c) = 0$. Let U be an open disc with $w \in U \subset A(w)$. Then $f^k(c) \notin U$ for all $k = 0, 1, 2, \ldots$ and all c such that $f'(c) = 0$. This enables us to choose, for each k, a branch of the inverse f^{-k} on U that is a continuous analytic function with $f^{-k}(w) = w$. If $z \in f^{-k}(U)$ then $f^k(z) \in U \subset A(w)$, so $z \in A(w)$; thus $f^{-k}(U) \subset A(w)$ for all k. Since $A(w)$ is a bounded subset of $\mathbb{C}$, Montel's theorem implies that $\{f^{-k}\}_{k=0}^{\infty}$ is a normal family on U. However, since w is a repelling fixed point of f^{-1}, no subsequence of $f^{-k}(z)$ can have a subsequence uniformly convergent to an analytic function near w , so $\{f^{-k}\}_{k=0}^{\infty}$ cannot be normal by the definition of a normal family. We conclude that $c \in A(w)$ for some c with $f'(c) = 0$.

14.18 Let $f(z) = az^2 + bz + d$ with $a \neq 0$. Then f^p is a polynomial of order 2^p. Suppose that f^p has an attractive fixed point w. By Exercise 14.17, the basin of attraction $A(w)$ of w under iteration of f^p contains a point c such that

$$\begin{aligned} 0 = (f^p)'(c) &= f'(c)f'(f(c)) \ldots f'(f^{p-1}(c)) \\ &= (2ac + b)(2af(c) + b) \ldots (2af^{p-1}(c) + b) \end{aligned}$$

using the chain rule. It follows that for some $0 \leq r \leq p - 1$ we have $0 = 2af^r(c) + b$, that is $f^r(c) = -b/2a$ for some c in $A(w)$.

If f has an attractive periodic orbit of order p then f^p has some attractive fixed point w, so $-b/2a$ is attracted to this periodic orbit under iteration of f. We conclude that there can be at most one attractive periodic orbit.

Chapter 15

15.1 This is a particular case of the random Cantor set described in Section 15.1. In this case

$$\begin{array}{ll} C_1 = C_2 = \dfrac{1}{3} & \text{with probability } \dfrac{1}{2} \\ C_1 = C_2 = \dfrac{1}{6} & \text{with probability } \dfrac{1}{2} \end{array}.$$

By Theorem 15.1, the Hausdorff dimension is given by the solution s of the expectation equation

$$1 = \mathsf{E}(C_1^s + C_2^s) = \frac{1}{2}(3^{-s} + 3^{-s}) + \frac{1}{2}(6^{-s} + 6^{-s}) = 3^{-s} + 6^{-s},$$

giving $s = 0.4895\ldots$.

15.2 This is a particular case of the random construction considered in Theorem 15.2. Since the number of segments at least doubles at each step, the probability of extinction occurring in the construction is 0. Writing C_1, C_2, C_3, C_4 for the length ratios of each of the four subsegments at each step, we have

$$\begin{array}{ll} C_1 = C_2 = C_3 = C_4 = \dfrac{1}{3} & \text{with probability } \dfrac{1}{2} \\ C_1 = C_4 = \dfrac{1}{3} \text{ and } C_2 = C_3 = 0 & \text{with probability } \dfrac{1}{2} \end{array}.$$

By Theorem 15.2, the Hausdorff dimension is given by the solution s of the expectation equation

$$1 = \mathsf{E}\left(\sum_{i=1}^{4} C_i^s\right) = \frac{1}{2}(4 \times 3^{-s}) + \frac{1}{2}(2 \times 3^{-s}) = 3 \times 3^{-s}.$$

Thus $s = 1$.

15.3 Let E_0 be the (closed) parallelogram with vertices $(0, 0)$, $(1/2, \sqrt{3}/6)$, $(1, 0)$ and $(1/2, -\sqrt{3}/6)$, so that E_0 has diameter 1. Let F be any 'random' von Koch curve constructed by substituting an upwards or downwards figure at each stage. Let E_1 be the set consisting of 4 similar parallelograms of diameter 3^{-1} with axes on the 4 segments of the first stage of the construction of F; all these parallelograms are contained in E_0. Let E_2 be the set consisting of similar parallelograms of diameter 3^{-2} with axes on the 4^2 segments of the second stage of the construction of F, each contained in a parallelogram of E_1, and so on. Thus E_k consists of 4^k parallelograms of diameters 3^{-k} with disjoint interiors, each contained in a parallelogram of E_{k-1}. Then $F = \bigcap_{k=0}^{\infty} E_k$. By Proposition 4.1, $\dim_{\mathrm{H}} F \leq \overline{\dim}_{\mathrm{B}} F \leq \lim_{k\to\infty} \log 4^k / -\log 3^{-k} = \log 4/\log 3$.

For the lower bound, let μ be the mass distribution obtained by repeated subdivision so that $\mu(P) = 4^{-k}$ for each parallelogram P of E_k. Then if $|U| < 1$, and k is the integer such that $2^{-1}3^{-k-1} \le |U| < 2^{-1}3^{-k}$, it is easy to see from the geometry of the parallelograms that U intersects at most 6 parallelograms of E_k. Thus

$$\mu(U) \le 6 \times 4^{-k} = 6 \times 3^{-k \log 4/\log 3} \le 6 \times (6|U|)^{\log 4/\log 3},$$

so it follows from the Mass distribution principle 4.2 that $\dim_{\mathrm{H}} F \ge \log 4/\log 3$.

15.4 This fits into the context of Theorem 15.2. The expected number of subtriangles at each stage of the construction is $3p$, so extinction of the construction, leading to the empty set, will occur if and only if $3p \le 1$, that is if and only if $p \le 1/3$.

If $p > 1/3$, we have $\mathsf{P}(N = j) = \begin{pmatrix} 3 \\ j \end{pmatrix} p^j (1-p)^{3-j}$, and by the binomial theorem, equation (15.8) becomes $(tp + (1-p))^3 = t$. By Theorem 15.2 the smallest positive solution t of this cubic equation gives the probability that F is empty.

To find the dimension of F when it is non-empty, write C_1, C_2, C_3 for the ratios of the three similarities, so that $C_i = \frac{1}{2}$ with probability p and $C_i = 0$ with probability $1-p$, independently for $i = 1, 2, 3$. Thus by (15.9), the Hausdorff dimension is given by the solution s of the expectation equation

$$1 = \mathsf{E}\left(\sum_{i=1}^{3} C_i^s\right) = 3(p2^{-s} + (1-p)0) = 3p2^{-s}.$$

Taking logarithms gives $\dim_{\mathrm{H}} F = s = \log 3p/\log 2$.

15.5 Let v be a vertex of a triangle T of the kth stage of the construction of the standard Sierpiński triangle. We claim that in the random Sierpiński triangle F, with probability 1 the set $F \cap T$ does not contain some neighbourhood of v. To see this, let $T \supset T_{k+1} \supset T_{k+2} \supset \dots$ be the nested triangles at the subsequent stages of the standard Sierpiński triangle construction that contain the vertex v. There is a probability p that each of these triangles is retained in the random construction, so the probability that $T_{k+1}, \dots, T_{k+n}$ are all retained is p^n. Using continuity of probabilities, since $p^n \to 0$ as $n \to \infty$, the probability that all the triangles $T_{k+1}, T_{k+2}, \dots$ are retained is 0. Thus with probability 1, one of the triangles $T_{k+1}, T_{k+2}, \dots$ is removed, so $(F \cap T) \subset (T \setminus T_{k+n})$ for some n.

Since there are countably many triangle vertices in the Sierpiński triangle construction, with probability 1 the complement of F contains a neighbourhood of every one of these vertices. Thus, if $x, y \in F$, let T be a triangle of the standard Sierpiński triangle construction with $x \in T$ and $y \notin T$. There is a neighbourhood of each of the three vertices of T that is not in F, and these neighbourhoods disconnect x from the parts of F outside T, so x and y are in different components of F.

15.6 Since the set F may be covered with either a single unit interval, or the aggregate of coverings of $F \cap I_1$ and $F \cap I_2$, we have that

$$\begin{aligned}\mathcal{H}^s_\infty(F) &\le \min\{1, \mathcal{H}^s_\infty(F \cap I_1) + \mathcal{H}^s_\infty(F \cap I_2)\}\\ &\le C_1^s \frac{\mathcal{H}^s_\infty(F_n I_1)}{C_1^s} + C_2^s \frac{\mathcal{H}^s_\infty(F_n I_2)}{C_2^s},\end{aligned}$$

Since $\mathcal{H}^s_\infty(F_n I_1)/C_1^s$ and $\mathcal{H}^s_\infty(F_n I_2)/C_2^s$ are independent realizations of $\mathcal{H}^s_\infty(F)$ and independent of $\{C_1, C_2\}$,

$$\begin{aligned}\mathsf{E}(\mathcal{H}^s_\infty(F)) &\le \mathsf{E}(C_1^s\mathcal{H}^s_\infty(F) + C_2^s\mathcal{H}^s_\infty(F))\\ &\le \mathsf{E}(C_1^s + C_2^s)\mathsf{E}(\mathcal{H}^s_\infty(F)) = \mathsf{E}(\mathcal{H}^s_\infty(F)).\end{aligned}$$

Thus equality holds, so since the terms are finite, either $\mathsf{E}(\mathcal{H}^s_\infty(F)) = 0$, or $\mathcal{H}^s_\infty(F) = C_1^s(\mathcal{H}^s_\infty(F_n I_1)/C_1^s) + C_2^s(\mathcal{H}^s_\infty(F_n I_2)/C_2^s)$ almost surely. In the latter case

$$\operatorname{esssup}\mathcal{H}^s_\infty(F) = \operatorname{esssup}(C_1^s + C_2^s)\operatorname{esssup}\mathcal{H}^s_\infty(F).$$

Hence either $\operatorname{esssup}(C_1^s + C_2^s) = 1$, which would imply that $C_1^s + C_2^s = 1$ almost surely, since $\mathsf{E}(C_1^s + C_2^s) = 1$, or else $\mathcal{H}^s_\infty(F) = 0$ almost surely. If $\mathcal{H}^s_\infty(F) = 0$ then given $\delta > 0$, by scaling, if $|I_{i_1,\dots,i_k}| \le b^k \le \delta$ then $\mathcal{H}^s_\delta(F \cap I_{i_1,\dots,i_k}) = 0$, so by taking unions of such basic intervals $\mathcal{H}^s_\delta(F) = 0$. Letting $\delta \to 0$ we conclude that $\mathcal{H}^s(F) = 0$.

15.7 Writing $C_{I,j} = |I_{i_1,\dots,i_k,j}|/|I_{i_1,\dots,i_k}|$ where $I = I_{i_1,\dots,i_k}$, we have that

$$\begin{aligned}\mathsf{E}\left(X_{k+1}^2|\mathcal{F}_k\right) &= \mathsf{E}\Bigg(\Bigg(\sum_{I\in E_{k+1}} |I|^s\Bigg)^2 |\mathcal{F}_k\Bigg)\\ &= \mathsf{E}\Bigg(\Bigg(\sum_{I\in E_k} |I|^s(C_{I,1}^s + C_{I,2}^s)\Bigg)^2\Bigg|\mathcal{F}_k\Bigg)\\ &= \mathsf{E}\Bigg(\sum_{I\neq J\in E_k} |I|^s|J|^s(C_{I,1}^s + C_{I,2}^s)(C_{J,1}^s + C_{J,2}^s)\\ &\qquad + \sum_{I\in E_k} |I|^{2s}(C_{I,1}^s + C_{I,2}^s)^2|\mathcal{F}_k\Bigg)\end{aligned}$$

$$= \sum_{I \neq J \in E_k} |I|^s |J|^s + \sum_{I \in E_k} |I|^{2s} \mathsf{E}\left((C_1^s + C_2^s)^2\right)$$

$$\leq \left(\sum_{I \in E_k} |I|^s\right)^2 + a \sum_{I \in E_k} |I|^{2s} = X_k^2 + a \sum_{I \in E_k} |I|^{2s},$$

where $a = \mathsf{E}((C_1^s + C_2^s)^2)$ and we have used (15.2). Taking unconditional expectations, we get

$$\mathsf{E}(X_{k+1}^2) \leq \mathsf{E}(X_k^2) + a\mathsf{E}\left(\sum_{I \in E_k} |I|^{2s}\right) = \mathsf{E}(X_k^2) + a(\mathsf{E}(C_1^{2s} + C_2^{2s}))^k$$

$$= \mathsf{E}(X_k^2) + a\gamma^k,$$

where $\gamma = \mathsf{E}(C_1^{2s} + C_2^{2s}) < 1$, using (15.3) repeatedly. Applying this $(k - 1)$ times,

$$\mathsf{E}(X_k^2) \leq \mathsf{E}(X_1^2) + a(\gamma + \gamma^2 + \ldots + \gamma^{k-1}) \leq \mathsf{E}(X_1^2) + a\gamma/(1 - \gamma)$$

for all k. Thus X_k^2 is an $\mathcal{L}^2$ bounded martingale.

15.8 Fix $p > p_0$, and let $r > 0$ be the probability that the random fractal F_p constructed by the percolation process is non-empty and therefore, with the same probability, contains a non-trivial connected component. Let I be a square that is retained at some kth level of the 3×3 fractal percolation construction. There is a probability $p(1 - p)^8$ that at the $(k + 1)$th stage of the construction the middle $(k + 1)$th level subsquare of I is selected and the other 8 sub-squares are removed, and by self-similarity there is a probability of r that this middle square intersects F_p in a non-trivial connected component. Thus there is a probability at least $s \equiv rp(1 - p)^8$ that a square I retained at the kth level contains a non-trivial connected component of F_p that does not extend outside I.

Given $\epsilon > 0$ and a probability s, the laws of large numbers imply that there is an integer N_0 such that, if $N \geq N_0$, in N independent trials each with probability s of success, there is a probability of at least $1 - \epsilon$ that at least $Ns/2$ of the trials will be successful. Thus, suppose that, for some k there are at least N squares in E_k. There is a probability of at least s that, independently, each of these squares contains a distinct non-trivial connected component, so there is a probability of at least $1 - \epsilon$ that at least $Ns/2$ of these squares contain a non-trivial connected component.

Finally, with r the probability of F_p being non-empty, given $\epsilon > 0$ and N, there is an integer k such that with probability at least $r - \epsilon$ the kth level stage of the construction, E_k, contains at least N squares. (This follows from extinction properties of branching processes.) Thus, given $\epsilon > 0$ there is a probability of at least $(r-\epsilon)(1-\epsilon) > r-2\epsilon$ that F_p contains at least $Ns/2$ distinct connected components. Since this is true for all $\epsilon > 0$ and $N \geq N_0$, the random set F_p contains infinitely many distinct connected components with probability r.

Chapter 16

16.1 Let $N_\delta(X[0,1])$ denote the (random) least number of sets of diameter δ that can cover the Brownian trail $X[0,1]$. Then for each n, since the Brownian trails $X[(i-1)2^{-n}, i2^{-n}]$ for $i = 1, 2, \ldots, 2^n$ have the same distribution as $X[0,1]$ but under a similarity scaling by a factor $2^{-n/2}$, we conclude that $N_{2^{-n/2}}(X[(i-1)2^{-n}, i2^{-n}])$ has the same statistical distribution as $N_1(X[0,1])$. In particular, taking expectations,

$$\mathsf{E}(N_{2^{-n/2}}(X[(i-1)2^{-n}, i2^{-n}])) = \mathsf{E}(N_1(X[0,1])),$$

so taking the aggregate of such coverings,

$$\mathsf{E}(N_{2^{-n/2}}(X[0,1])) \leq 2^n \mathsf{E}(N_1(X[0,1])).$$

Thus, for all $\epsilon > 0$, we have

$$\mathsf{E}(2^{-n(1+\epsilon)} N_{2^{-n/2}}(X[0,1])) \leq 2^{-n\epsilon} \mathsf{E}(N_1(X[0,1])).$$

Summing,

$$\mathsf{E}\left(\sum_{n=1}^{\infty} 2^{-n(1+\epsilon)} N_{2^{-n/2}}(X[0,1])\right) \leq \sum_{n=1}^{\infty} 2^{-n\epsilon} \mathsf{E}(N_1(X[0,1])) < \infty.$$

Thus, with probability 1, $\sum_{n=1}^{\infty} 2^{-n(1+\epsilon)} N_{2^{-n/2}}(X[0,1]) < \infty$, implying that for some random number C we have $N_{2^{-n/2}}(X[0,1]) \leq C 2^{n(1+\epsilon)}$ for all n, and so by Proposition 4.1 that $\overline{\dim}_{\mathrm{B}} X[0,1] \leq 2(1+\epsilon)$ for all $\epsilon > 0$. We conclude that $\overline{\dim}_{\mathrm{B}} X[0,1] \leq 2$.

16.2 This is a variation on the proof of Theorem 16.2. Consider Brownian motion $X : [0,1] \to \mathbb{R}^3$. Let $0 < \lambda < 1/2$. By an obvious modification of Proposition 16.1, there is with probability 1 a (random) number B such that

$$|X(t) - X(u)| \leq B|t-u|^{\lambda} \quad (t, u \in [0,1]),$$

so by Proposition 2.3 $\dim_{\mathrm{H}} X(F) \leq (1/\lambda)\dim_{\mathrm{H}} F = \log 2/\lambda \log 3$. This is true for all $0 < \lambda < 1/2$, so $\dim_{\mathrm{H}} X(F) \leq 2\log 2/\log 3 = \log 4/\log 3$.

For the lower bound, we define a measure by transferring Hausdorff measure on the Cantor set F to the trail. With $q = \log 2/\log 3$, define a random measure μ on $X(F)$ by $\mu(A) = \mathcal{H}^q\{t : t \in F \text{ and } X(t) \in A\}$ for $A \subset \mathbb{R}^3$.

Thus for a function g on $\mathbb{R}^3$ we have $\int g(x)d\mu(x) = \int_F g(X(t))d\mathcal{H}^q(t)$. Then for $s < 2\log 2/\log 3$

$$\mathsf{E}\left(\iint |x-y|^{-s}d\mu(x)d\mu(y)\right) = \mathsf{E}\left(\int_F\int_F |X(t)-X(u)|^{-s}dtdu\right)$$
$$= \int_F\int_F \mathsf{E}(|X(t)-X(u)|^{-s})dtdu$$
$$= c_1\int_F\int_F |t-u|^{-s/2}dtdu < \infty$$

using Exercise 4.11, with c_1 as in (16.8). It follows from Theorem 4.13(a) that $\dim_{\mathrm{H}}X(F) \ge s$ for all $s < 2\log 2/\log 3$, so $\dim_{\mathrm{H}}X(F) \ge 2\log 2/\log 3 = \log 4/\log 3$.

If X is index-α fractional Brownian motion, we get, in a similar way, that $\dim_{\mathrm{H}}X(F) = \log 2/\alpha\log 3$, using that, with probability 1, X satisfies a Hölder condition of index λ for all $0 < \lambda < \alpha$ (Proposition 16.6). For the lower bound we replace (16.8) by $\mathsf{E}(|X(t+h)-X(t)|^{-s}) = c_1 h^{-s\alpha}$.

16.3 The approach here is similar to that in Theorem 16.3. Suppose, for a contradiction, that $X[0,1]\cap F = \emptyset$ with probability 1. Using the isotropy and scaling of Brownian trails in $\mathbb{R}^3$, it follows that for every similarity σ, the probability that $\sigma(X[0,1])\cap F = \emptyset$ is also 1. By Fubini's theorem, with probability 1 we have that $\sigma(X[0,1])\cap F = \emptyset$ for almost all similarities σ. But with, probability 1, $\dim_{\mathrm{H}}X[0,1] = 2$, so since $\dim_{\mathrm{H}}X[0,1] + \dim_{\mathrm{H}}F - 3 > 2+1-3 = 0$, this contradicts Theorem 8.2(a). We conclude that $X[0,1]$ intersects F with positive probability.

16.4 The easiest way to see this is to note that the affine transformation $f:\mathbb{R}^2\to\mathbb{R}^2$ given by $f(t,u) = (t, u+ct)$ is bi-Lipschitz (since the matrix $\begin{pmatrix} 1 & 0 \\ c & 1\end{pmatrix}$ is invertible). Then $f(\mathrm{graph}X(t)) = \mathrm{graph}(X(t)+ct)$, so $\dim_{\mathrm{H}}(\mathrm{graph}(X(t)+ct)) = \dim_{\mathrm{H}}(\mathrm{graph}X(t)) = 1\frac{1}{2}$ almost surely, using Corollary 2.4 and Theorem 16.4.

16.5 Let $t = t_0 < t_1 < \dots < t_n = u$. Then $X(t_{i-1}) \le X(t_i)$ with probability $\frac{1}{2}$, independently for $i = 1, 2, \dots, n$. Hence the probability that $X(t_{i-1}) \le X(t_i)$ for all $i = 1, 2, \dots, n$ is 2^{-n} and similarly the probability that $X(t_{i-1}) \ge X(t_i)$ for all $i = 1, 2, \dots, n$ is 2^{-n}. If $X(t)$ is monotonic on $[t,u]$ then one of these possibilities must occur, so $\mathsf{P}(X(t)$ is monotonic on $[t,u]) \le 2\times 2^{-n}$. This is true for all positive integers n, so $\mathsf{P}(X(t)$ is monotonic on $[t,u]) = 0$.

Since there are countably many rational numbers, there are countably many 'rational intervals', i.e. intervals with rational endpoints. Since a countable union of events each of probability 0 has probability 0, we conclude that $\mathsf{P}(X(t)$ is monotonic on some rational interval $= 0$. Since every non-degenerate interval contains a rational interval, this implies that $\mathsf{P}(X(t)$ is monotonic on some interval $= 0$.

16.6 First we show that there is a number $\gamma > 0$ such that, given $X[0, 1]$, the probability that $X(t) = 0$ for some $t > 1$ is at least γ, i.e.

$$\mathsf{P}(X(t) = 0 \text{ for some } t > 1 | X(t)\ (0 \le t \le 1)) \ge \gamma. \tag{1}$$

To see this, suppose that $X(1) = -M < 0$. Then, conditional on this, using (16.2),

$$\begin{aligned}\mathsf{P}(X(t) = 0 \text{ for some } t > 1) &\ge \mathsf{P}(X(1 + M^2) - X(1) > M)\\ &= \frac{1}{M\sqrt{2\pi}} \int_M^\infty \exp\left(\frac{-x^2}{2M^2}\right) dx\\ &\ge \frac{1}{M\sqrt{2\pi}} \int_M^{2M} \exp(-2) dx\\ &= \frac{\exp(-2)}{\sqrt{2\pi}} \equiv \gamma,\end{aligned}$$

with similar estimates if $X(1) = M \ge 0$. In particular, it follows from (1) that

$$\mathsf{P}(X(t) = 0 \text{ for some } t > 1 | X(t) \ne 0\ (0 < t \le 1)) \ge \gamma.$$

Now set $p = \mathsf{P}(X(t) = 0 \text{ for some } 0 < t \le 1)$. By statistical self-similarity of Brownian motion, for every $N > 0$,

$$p = \mathsf{P}(X(t) = 0 \text{ for some } 0 < t \le N\}, \tag{2}$$

so taking a union over all positive integers N, and noting that these are decreasing events (i.e. if $N_1 < N_2$ and $X(t) \ne 0$ for $0 < t \le N_2$ then $X(t) \ne 0$ for $0 < t \le N_1$), we conclude that $p = \mathsf{P}(X(t) = 0 \text{ for some } t > 0)$. But

$$\begin{aligned}p &= \mathsf{P}(X(t) = 0 \text{ for some } t > 0)\\ &\ge \mathsf{P}(X(t) = 0 \text{ for some } 0 < t \le 1)\\ &\qquad + \mathsf{P}(X(t) \ne 0 \text{ for } 0 < t \le 1 \text{ and }\\ &\qquad X(t) = 0 \text{ for some } t > 1)\\ &= p + \mathsf{P}(X(t) = 0 \text{ for some } t > 1 | X(t) \ne 0\ (0 < t \le 1))\\ &\qquad \times \mathsf{P}(X(t) \ne 0\ (0 < t \le 1))\\ &\ge p + \gamma(1 - p)\end{aligned}$$

using the definition of conditional probability (1.15). Thus $p \ge p + \gamma(1 - p)$ which requires $1 - p = 0$, that is $p = 1$. Thus with probability 1, $X(t) = 0$ for some $t > 0$.

By (2), with probability 1, $X(t) = 0$ for some $0 < t \le 1/n$, for all $n = 1, 2, \ldots$. This can only happen if $X(t) = 0$ infinitely often in every interval $(0, a)$ with $a > 0$.

16.7 Let X be Brownian motion. As in (16.4), with $h > 0$,

$$p(r) \equiv \mathsf{P}(0 \le X(t+h) - X(t) \le r) = \frac{1}{\sqrt{2\pi h}} \int_0^r \exp\left(\frac{-x^2}{2h}\right) dx.$$

Thus

$$\begin{aligned}\mathsf{E}(|X(t+h) - X(t)|^q) &= 2\int_0^\infty r^q dp(r) \\ &= \frac{2}{\sqrt{2\pi h}} \int_0^\infty r^q \exp\left(\frac{-r^2}{2h}\right) dr \\ &= \frac{2}{\sqrt{2\pi}} h^{q/2} \int_0^\infty u^q \exp\left(\frac{-u^2}{2}\right) du, \\ &= ch^{q/2}\end{aligned}$$

on substituting $u = r/\sqrt{h}$, as required.

16.8 Let $\lambda > \alpha$. Suppose that, for a given t and b, there almost surely exists H_0 such that

$$|X(t+h) - X(t)| \le b|h|^\lambda \text{ for all } |h| \le H_0. \tag{1}$$

Then, by Egoroff's theorem, there exists $h_0 > 0$ such that with probability at least $1/2$ we have $|X(t+h) - X(t)| \le b|h|^\lambda$ for all $|h| \le h_0$.

On the other hand, from (16.10), we have

$$\begin{aligned}\mathsf{P}(|X(t+h) - X(t)| \le b|h|^\lambda) &= \frac{2}{\sqrt{2\pi}} |h|^{-\alpha} \int_0^{b|h|^\lambda} \exp\left(\frac{-u^2}{2|h|^{2\alpha}}\right) du \\ &\le \frac{2}{\sqrt{2\pi}} |h|^{-\alpha} \int_0^{b|h|^\lambda} 1du \\ &= \frac{2}{\sqrt{2\pi}} b|h|^{\lambda-\alpha},\end{aligned}$$

so by taking h sufficiently small we get a contradiction.

We conclude that for all t, with probability 1 there is no number H_0 such that (1) holds. Thus, by Fubini's theorem, with probability 1, there is, for almost all t, no H_0 such that (1) holds.

16.9 Write $X(t) = (X_1(t), X_2(t))$, where X_1 and X_2 are independent index-α_1 and index-α_2 fractional Brownian motions, with $1/2 \le \alpha_1 \le \alpha_2 < 1$. To get an upper bound for the dimension of $X[0,1]$ we use the Hölder estimate of Proposition 16.6. Thus, given $\epsilon > 0$, there are, with probability 1, (random) constants $0 < B_1, B_2 < \infty$ such that for $t, u \in [0,1]$

$$|X_1(t) - X_1(u)| \le B_1|t-u|^{\alpha_1-\epsilon},$$
$$|X_2(t) - X_2(u)| \le B_2|t-u|^{\alpha_2-\epsilon}.$$

Thus, for $k = 1, 2, \ldots$, if $|t-u| \le 2^{-k}$ then $X[t,u]$ is contained in a rectangle with sides $B_1 2^{-k(\alpha_1-\epsilon)}$ and $B_2 2^{-k(\alpha_2-\epsilon)}$. Dividing this rectangle into approximate squares of sides at most $B_2 2^{-k(\alpha_2-\epsilon)}$, the rectangle, and thus $X[t,u]$, may be covered by $2B_1 2^{-k(\alpha_1-\epsilon)}/B_2 2^{-k(\alpha_2-\epsilon)} = (2B_1/B_2)2^{k(\alpha_2-\alpha_1)}$ sets of diameter at most $2B_2 2^{-k(\alpha_2-\epsilon)}$, provided k is sufficiently large. Thus, dividing the interval $[0,1]$ into 2^k subintervals of length 2^{-k}, the trail $X[0,1]$ may be covered by $(2B_1/B_2)2^{k(1+\alpha_2-\alpha_1)}$ sets of diameter at most $2B_2 2^{-k(\alpha_2-\epsilon)}$. It follows by Proposition 4.1 that

$$\begin{aligned}\dim_{\mathrm{H}} X[0,1] \le \overline{\dim}_{\mathrm{B}} X[0,1] &\le \lim_{k\to\infty} \frac{\log((2B_1/B_2)2^{k(1+\alpha_2-\alpha_1)})}{-\log(2B_2 2^{-k(\alpha_2-\epsilon)})}\\ &= \frac{1+\alpha_2-\alpha_1}{\alpha_2-\epsilon},\end{aligned}$$

for all $\epsilon > 0$, so $\dim_{\mathrm{H}} X[0,1] \le (1+\alpha_2-\alpha_1)/\alpha_2$.

For the lower bound we use the potential theoretic method. We need to estimate the integral $\mathsf{E}((|X_1(t+h) - X_1(t)|^2 + |X_2(t+h) - X_2(t)|^2)^{-s/2})$, so we first consider the X_1 part. Write

$$\begin{aligned}p(r) &= \mathsf{P}(0 \le X_1(t+h) - X_1(t) \le r)\\ &= \frac{1}{h^{\alpha_1}(2\pi)^{1/2}} \int_0^r \exp(-x^2/2h^{2\alpha_1})dx, \qquad (1)\end{aligned}$$

by (16.10). For fixed y,

$$\begin{aligned}&\mathsf{E}((|X_1(t+h) - X_1(t)|^2 + y^2)^{-s/2})\\ &= 2\int_0^\infty (r^2+y^2)^{-s/2} dp(r)\\ &= ch^{-\alpha_1}\int_0^\infty (r^2+y^2)^{-s/2}\exp(-r^2/2h^{2\alpha_1})dr\\ &= c\int_0^\infty (u^2h^{2\alpha_1}+y^2)^{-s/2}\exp(-u^2/2)du\\ &\le c\int_0^{y/h^{\alpha_1}} y^{-s}\exp(-u^2/2)du + c\int_{y/h^{\alpha_1}}^\infty (uh^{\alpha_1})^{-s}\exp(-u^2/2)du\end{aligned}$$

$$\leq c\int_0^{y/h^{\alpha_1}} y^{-s}du + c\int_{y/h^{\alpha_1}}^{\infty} (uh^{\alpha_1})^{-s}du$$
$$\leq c_1 y^{1-s}h^{-\alpha_1}$$

where c, c_1 do not depend on h, and we have substituted $u = r/h^{\alpha_1}$.

Using the analogue of (1) for index-α_2 fractional Brownian motion, we get in a similar way,

$$\begin{aligned}
\mathsf{E}((|X_1(t+h) - X_1(t)|^2 &+ |X_2(t+h) - X_2(t)|^2)^{-s/2}) \\
&= \mathsf{E}((c_1|X_2(t+h) - X_2(t)|^{1-s}h^{-\alpha_1}) \\
&= c_2 h^{-\alpha_2}\int_0^{\infty} y^{1-s}h^{-\alpha_1}\exp(-y^2/2h^{2\alpha_2})dy \\
&= c_2 h^{-\alpha_1+(1-s)\alpha_2}\int_0^{\infty} u^{1-s}\exp(-u^2/2)du \\
&= c_3 h^{-\alpha_1+(1-s)\alpha_2}
\end{aligned}$$

where we have substituted $u = y/h^{\alpha_2}$.

Define a random measure on $X[0,1]$ by $\mu(A) = \mathcal{L}\{t : 0 \leq t \leq 1 \text{ and } X(t) \in A\}$, so that for $f : \mathbb{R}^2 \to \mathbb{R}$, $\int f d\mu = \int_0^1 f(X(t))dt$. Then

$$\begin{aligned}
&\mathsf{E}\left(\iint |x-y|^{-s}d\mu(x)d\mu(y)\right) \\
&= \mathsf{E}\left(\int_0^1\int_0^1 (|X_1(t) - X_1(u)|^2 + |X_2(t) - X_2(u)|^2)^{-s/2})dt\,du\right) \\
&= \int_0^1\int_0^1 \mathsf{E}((|X_1(t) - X_1(u)|^2 + |X_2(t) - X_2(u)|^2)^{-s/2}))dt\,du \\
&\leq \int_0^1\int_0^1 c_3|t-u|^{-\alpha_1+(1-s)\alpha_2}dt\,du.
\end{aligned}$$

This is finite if $-\alpha_1 + (1-s)\alpha_2 > -1$, that is if $s < (\alpha_2 - \alpha_1 + 1)/\alpha_2$. Thus if $s < (\alpha_2 - \alpha_1 + 1)/\alpha_2$ then almost surely the trail $X[0,1]$ supports a measure μ with finite s-energy, so by Proposition 4.13, $\dim_{\mathrm{H}} X[0,1] \leq (\alpha_2 - \alpha_1 + 1)/\alpha_2$ almost surely.

16.10 We take as our starting point that for index-α fractional Brownian motion $\mathsf{E}(X(t)^2) = |t|^{2\alpha}$ and $\mathsf{E}((X(t) - X(u))^2) = |t-u|^{2\alpha}$. Expanding the latter,

$$\begin{aligned}
|t-u|^{2\alpha} &= \mathsf{E}(X(t)^2) + \mathsf{E}(X(u)^2) - 2\mathsf{E}(X(t)X(u)) \\
&= |t|^{2\alpha} + |u|^{2\alpha} - 2\mathsf{E}(X(t)X(u))
\end{aligned}$$

so
$$2\mathsf{E}(X(t)X(u)) = |t|^{2\alpha} + |u|^{2\alpha} - |t-u|^{2\alpha}.$$

Thus
$$\begin{aligned}
&2\mathsf{E}((X(t)-X(0))(X(t+h)-X(t)))\\
&= 2(\mathsf{E}(X(t)X(t+h)) - \mathsf{E}(X(0)X(t+h)) - \mathsf{E}(X(t)X(t)) + \mathsf{E}(X(0)X(t)))\\
&= (|t|^{2\alpha} + |t+h|^{2\alpha} - |h|^{2\alpha}) - 0 - (2|t|^{2\alpha}) + 0\\
&= |t+h|^{2\alpha} - |t|^{2\alpha} - |h|^{2\alpha},
\end{aligned}$$

giving
$$\mathsf{E}((X(t)-X(0))(X(t+h)-X(t))) = \frac{1}{2}(|t+h|^{2\alpha} - |t|^{2\alpha} - |h|^{2\alpha}).$$

For $t, h \neq 0$, we have, by elementary calculus, that if $\frac{1}{2} < \alpha < 1$ then $|t+h|^{2\alpha} > |t|^{2\alpha} + |h|^{2\alpha}$, so $\mathsf{E}((X(t)-X(0))(X(t+h)-X(t))) > 0$, and the increments $X(t)-X(0)$ and $X(t+h)-X(t)$ are positively correlated. Thus if the sample path has increased after a certain time, there is a tendency for it to continue to increase, and if it has decreased there is a tendency for it to decrease further.

Similarly, if $0 < \alpha < \frac{1}{2}$ then $|t+h|^{2\alpha} < |t|^{2\alpha} + |h|^{2\alpha}$, so $\mathsf{E}((X(t)-X(0))(X(t+h)-X(t))) < 0$, and the increments $X(t)-X(0)$ and $X(t+h)-X(t)$ are negatively correlated. Thus if the sample path has increased after a certain time, there is a tendency for it to decrease.

Chapter 17

17.1 The Legendre transform is $\inf_q\{e^{-q} + q\alpha\}$. Writing $g(q) = e^{-q} + q\alpha$ we have
$$\frac{dg}{dq} = -e^{-q} + \alpha, \qquad \frac{d^2g}{dq^2} = e^{-q},$$
so g takes a minimum at $q = -\log\alpha$, so the Legendre transform is $e^{\log\alpha} - \alpha\log\alpha = \alpha(1-\log\alpha)$.

17.2 If $x \in \mathrm{spt}\mu_1$ then $\mu_2(B(x,r)) = 0$ is r is small enough, since the supports of μ_1 and μ_2 are disjoint, so $\nu(B(x,r) = \mu_1(B(x,r))$ for small r, giving $\dim_{\mathrm{loc}}\nu(x) = \dim_{\mathrm{loc}}\mu_1(x)$. Similarly, if $x \in \mathrm{spt}\mu_2$ then $\dim_{\mathrm{loc}}\nu(x) = \dim_{\mathrm{loc}}\mu_2(x)$. Thus
$$\{x : \dim_{\mathrm{loc}}\nu(x) = \alpha\} = \{x : \dim_{\mathrm{loc}}\mu_1(x) = \alpha\} \cup \{x : \dim_{\mathrm{loc}}\mu_2(x) = \alpha\}.$$

We get the fine spectra by taking the Hausdorff dimensions of these sets, so
$$f_{\mathrm{H}}^{\nu}(\alpha) = \max\{f_{\mathrm{H}}^{1}(\alpha), f_{\mathrm{H}}^{2}(\alpha)\}.$$

Taking μ_1 and μ_2 to be self-similar measures, such that f_{H}^1 and f_{H}^2 are different concave functions with graphs that cross, f_{H}^{ν} will fail to be a convex function.

17.3 Suppose that $b|x-y| \le |g(x)-g(y)| \le c|x-y|$ where $0<b\le c$. Then for $x\in\mathbb{R}^n$ we have $g(B(x,r/c))\subset B(g(x),r)\subset g(B(x,r/b))$, or $B(x,r/c)\subset g^{-1}(B(g(x),r))\subset B(x,r/b)$. Thus $\mu(B(x,r/c))\le \nu(B(g(x),r))\le \mu(B(x,r/b))$. Taking logarithms, for small enough r.

$$\frac{\log\mu(B(x,r/c))}{\log r/c+\log c}\ge\frac{\log\nu(B(g(x),r))}{\log r}\ge\frac{\log\mu(B(x,r/b))}{\log r/b+\log b},$$

so letting $r\to 0$, we get $\dim_{\mathrm{loc}}\mu(x)\ge\dim_{\mathrm{loc}}\nu(g(x))\ge\dim_{\mathrm{loc}}\mu(x)$ assuming these limits exist, so $\dim_{\mathrm{loc}}\nu(g(x))=\dim_{\mathrm{loc}}\mu(x)$.

It follows that for all α

$$g\{x:\dim_{\mathrm{loc}}\mu(x)=\alpha\}=\{y:\dim_{\mathrm{loc}}\nu(y)=\alpha\};$$

since g is bi-Lipschitz the sets $\{x:\dim_{\mathrm{loc}}\mu(x)=\alpha\}$ and $\{y:\dim_{\mathrm{loc}}\nu(y)=\alpha\}$ have the same dimension, that is the fine (Hausdorff) multifractal spectra for μ and ν are identical.

17.4 We have

$$1=p_1^q r_1^{\beta(q)}+p_2^q r_2^{\beta(q)}=(p_1^q+p_2^q)4^{-\beta(q)}.$$

Taking logarithms,

$$\beta(q)=\frac{\log(p_1^q+p_2^q)}{\log 4}.$$

For each q,

$$\alpha=-\frac{d\beta}{dq}=-\frac{p_1^q\log p_1+p_2^q\log p_2}{(p_1^q+p_2^q)\log 4},$$

so

$$f(\alpha)=-q\frac{d\beta}{dq}+\beta(q)=-\frac{q(p_1^q\log p_1+p_2^q\log p_2)}{(p_1^q+p_2^q)\log 4}+\frac{\log(p_1^q+p_2^q)}{\log 4}.$$

17.5 First, take $r=4^{-k}$. With the intervals $I_{i_1,\dots,i_k}$ in the construction of the middle half Cantor F set indexed in the usual way, see (17.22), we get from (17.6):

$$M_{4^{-k}}(q)=\sum_{i_1,\dots,i_k}\mu(I_{i_1,\dots,i_k})^q=\sum_{i_1,\dots,i_k}p_{i_1}^q\cdots p_{i_k}^q=(p_1^q+p_2^q)^k.$$

Now suppose that $4^{-k-1} \le r < 4^{-k}$. Then each mesh interval of length r intersects at most one of the k-th level component intervals $I_{i_1,\dots,i_k}$ of F, and each $I_{i_1,\dots,i_k}$ intersects at most 3 mesh intervals of length r. Thus

$$3^{-q}(p_1^q + p_2^q)^k = \sum_{i_1,\dots,i_k} \left(\frac{1}{3}\mu(I_{i_1,\dots,i_k})\right)^q \le M_r(q)$$
$$\le 3 \sum_{i_1,\dots,i_k} \mu(I_{i_1,\dots,i_k})^q = 3(p_1^q + p_2^q)^k.$$

Hence

$$\frac{\log 3^{-q}(p_1^q + p_2^q)^k}{-\log 4^{-k-1}} \le \frac{\log M_r(q)}{-\log r} \le \frac{\log 3(p_1^q + p_2^q)^k}{-\log 4^{-k}}$$

or

$$\frac{-q\log 3 + k\log(p_1^q + p_2^q)}{(k+1)\log 4} \le \frac{\log M_r(q)}{-\log r} \le \frac{\log 3 + k\log(p_1^q + p_2^q)}{k\log 4}.$$

Letting $r \to 0$, so $k \to \infty$, and (17.7) gives $\beta(q) = \lim_{r\to 0} \dfrac{\log M_r(q)}{-\log r}$ $= \dfrac{\log(p_1^q + p_2^q)}{\log 4}$.

17.6 By (17.26), $p_1^q(\frac{1}{2})^{-\beta} + p_2^q(\frac{1}{4})^{-\beta} = 1$. This is a quadratic equation in $x = (\frac{1}{2})^{-\beta}$, that is $p_2^q x^2 + p_1^q x - 1 = 0$. Thus $x = (-p_1^q + (p_1^{2q} + 4p_2^q)^{1/2})/2p_2^q$ (taking the positive solution since $x > 0$). Hence

$$\beta(q) = -\frac{\log x}{\log 2} = \frac{\log(2p_2^q) - \log((p_1^{2q} + 4p_2^q)^{1/2}) - p_1^q)}{\log 2}.$$

17.7 From (17.13)

$$\beta(q) - \beta(-q) = \frac{\log(p_1^q + p_2^q) - \log(p_1^{-q} + p_2^{-q})}{\log 3}$$
$$= \frac{\log p_1^q p_2^q (p_1^{-q} + p_2^{-q}) - \log(p_1^{-q} + p_2^{-q})}{\log 3}$$
$$= \frac{\log p_1^q p_2^q}{\log 3} = \frac{q\log(p_1 p_2)}{\log 3}.$$

Thus

$$
\begin{aligned}
f(\alpha) &= \inf_q\{\beta(q) + q\alpha\} = \inf_q \left\{\beta(-q) + q\left(\alpha + \frac{\log(p_1 p_2)}{\log 3}\right)\right\} \\
&= \inf_q \left\{\beta(q) - q\left(\alpha + \frac{\log(p_1 p_2)}{\log 3}\right)\right\} \\
&= f\left(-\alpha - \frac{\log(p_1 p_2)}{\log 3}\right).
\end{aligned}
$$

17.8 Since $p_1 < p_2$,

$$
\begin{aligned}
\beta(q) = \frac{\log(p_1^q + p_2^q)}{\log 3} &= \frac{\log p_2^q(1 + (p_1/p_2)^q)}{\log 3} \\
&= \frac{q\log p_2 + \log(1 + (p_1/p_2)^q)}{\log 3} \\
&= \frac{q\log p_2}{\log 3} + O((p_1/p_2)^q)) = \frac{q\log p_2}{\log 3} + o(1)
\end{aligned}
$$

as $q \to \infty$. Similarly

$$
\beta(q) = \frac{q\log p_1}{\log 3} + o(1)
$$

as $q \to -\infty$.

Thus if the β curve approaches the line $\beta = aq + b$ as $q \to \infty$, $a = \log p_2/\log 3$ and $b = 0$. Thus $\beta = q\log p_2/\log 3$ is the asymptote as $q \to \infty$, and similarly $\beta = q\log p_1/\log 3$ is the asymptote as $q \to -\infty$, both of these lines passing through the origin.

The slopes of the asymptotes give the extreme values of α, so $\alpha_{\min} = -\log p_2/\log 3$ and $\alpha_{\max} = -\log p_1/\log 3$. Moreover, $f(\alpha_{\min}) = f(\alpha_{\max}) = 0$, since these values are given by the intercepts of the asymptotes with the vertical axis.

17.9 From (17.34) $df/d\alpha = q$, so $d^2f/d\alpha^2 = dq/d\alpha = 1/(d\alpha/dq) = 1/(-d^2\beta/dq^2) < 0$, since $\beta(q)$ is convex, using that $\alpha = -d\beta/dq$.

17.10 Clearly, $\beta(1) = 0$. For $0 < q < 1$, we have, by Hölder's inequality, that

$$
\sum_{i=1}^m p_i^q r_i^{1-q} \le \left(\sum_{i=1}^m p_i\right)^q \left(\sum_{i=1}^m r_i\right)^{1-q} < 1.
$$

Since $\sum_{i=1}^m p_i^q r_i^{\beta(q)} = 1$, we have $\beta(q) < 1 - q$.

For $q > 1$, by Hölder's inequality

$$1 = \sum_{i=1}^{m} p_i = \sum_{i=1}^{m} (p_i r_i^{(1-q)/q})(r_i^{(q-1)/q})$$
$$\leq \left(\sum_{i=1}^{m} p_i^q r_i^{1-q}\right)^{1/q} \left(\sum_{i=1}^{m} r_i\right)^{(q-1)/q}.$$

Hence $1 < \sum_{i=1}^{m} p_i^q r_i^{1-q}$, so $\beta(q) > 1 - q$.

17.11 Note that for all $x \in \mathbb{R}^2$ and $r > 0$ we have $\mathrm{proj}(B(x,r)) = B_L(\mathrm{proj}x, r)$, so

$$(\mathrm{proj}\mu)(B_L(\mathrm{proj}x, r)) = \mu\{y \in \mathbb{R}^2 : \mathrm{proj}y \in B_L(\mathrm{proj}x, r)\} \geq \mu(B(x,r)).$$

Thus

$$\overline{\lim}_{r\to 0} \frac{\log((\mathrm{proj}\mu)(B_L(\mathrm{proj}x, r)))}{\log r} \leq \overline{\lim}_{r\to 0} \frac{\log \mu(B(x,r))}{\log r}.$$

17.12 Take $\epsilon > 0$. Let $\mathcal{Q}_k$ denote those kth level sequences $\mathbf{i} = (i_1, \dots, i_k) \in \mathcal{I}_k$ such that $\mu(I_{\mathbf{i}}) \leq |I_{\mathbf{i}}|^{\alpha-\epsilon}$. For $q < 0$:

$$\sum_{\mathbf{i}\in\mathcal{Q}_k} |I_{\mathbf{i}}|^{\beta+q(\alpha-\epsilon)} \leq \sum_{\mathbf{i}\in\mathcal{Q}_k} |I_{\mathbf{i}}|^{\beta} \mu(I_{\mathbf{i}})^q \leq \sum_{\mathbf{i}\in\mathcal{I}_k} |I_{\mathbf{i}}|^{\beta} \mu(I_{\mathbf{i}})^q$$
$$= \sum_{i_1,\dots,i_k} (c_{i_1} c_{i_2} \dots c_{i_k})^{\beta} (p_{i_1} p_{i_2} \dots p_{i_k})^q$$
$$= \left(\sum_{i=1}^{m} p_i^q c_i^{\beta}\right)^k = 1,$$

using a multinomial expansion and (17.26).

For each integer K, write

$$F_\alpha^K = \{x \in F : \mu(I_k(x)) \leq |I_k(x)|^{\alpha-\epsilon} \text{ for all } k \geq K\},$$

where $I_k(x)$ is the kth level interval containing x. Then for all $k \geq K$, the set $F^K \subset \bigcup_{\mathbf{i}\in\mathcal{Q}_k} I_{\mathbf{i}}$, so $\mathcal{H}^{\beta+q(\alpha-\epsilon)}_{c^k}(F^K) \leq 1$, since for a kth level interval, $|I_{\mathbf{i}}| \leq c^k$ where $c = \max_{1\leq i\leq m} c_i$. Letting $k \to \infty$ gives $\mathcal{H}^{\beta+q(\alpha-\epsilon)}(F^K) \leq 1$, so that $\dim_{\mathrm{H}}(F^K) \leq \beta + q(\alpha - \epsilon)$. But $F_\alpha \subset \bigcup_{K=1}^{\infty} F^K$, since if $\log \mu(I_k(x))/\log |I_k(x)| \to \alpha$ then $\mu(I_k(x)) \leq |I_k(x)|^{\alpha-\epsilon}$ for all k sufficiently large. Thus $\dim_{\mathrm{H}}(F_\alpha) \leq \beta + q(\alpha - \epsilon)$ for all $\epsilon > 0$, giving $\dim_{\mathrm{H}}(F_\alpha) \leq \beta + q\alpha$.

17.13 In the partial proof of Theorem 17.4, if we set

$$F^K = \{x \in F : \mu(I_k(x)) \geq |I_k(x)|^{\alpha+\epsilon} \text{ for some } k \geq K\}$$

then, as before, $\dim_{\mathrm{H}} \bigcup_{K=1}^{\infty} F^K \leq \beta + q(\alpha + \epsilon)$. But

$$S \equiv \{x \in F : \lim_{k\to\infty} \log \mu(I_k(x))/\log |I_k(x)| \leq \alpha\} \subset \bigcup_{K=1}^{\infty} F^K,$$

so $\dim_{\mathrm{H}} S \leq \beta + q(\alpha + \epsilon)$ for all $\epsilon > 0$, so $\dim_{\mathrm{H}} S \leq \beta + q\alpha$.

Chapter 18

18.1 From the way that densities of s-dimensional sets behave, see Chapter 5, we might heuristically expect that $\mu(B(0,r)) \sim cr^s$ where μ is the mass of the deposit and c is a constant. Thus the number of shaded squares k might be considered proportional to this mass $\sim c_1 r^s$, so $r \sim c_2 k^{1/s}$.

18.2 Again arguing heuristically, a rate of mass deposition proportional to $r(t)$ implies that

$$\frac{dm}{dt} = cr(t) \sim c_0 m^{1/s}$$

using $m(t) \sim cr(t)^s$. Thus, $m^{-1/s}\dfrac{dm}{dt} \sim c_0$, so solving this differential equation with $m(0) = 0$ gives $m^{1-1/s} \sim c_0 t$, so $m(t) \sim c_1 t^{s/(1-s)}$, and by Exercise 18.1, $r(t) \sim c_2 m(t)^{1/s} \sim t^{1/(1-s)}$.

18.3 We may express $u(y, t + h)$ in terms of $u(y, t)$ by

$$u(y, t + h) = \frac{1}{2\pi h} \int \exp\left(\frac{-(x - y)^2}{2h}\right) u(x, t)dx.$$

Differentiating with respect to h

$$\frac{\partial u}{\partial h} = \frac{1}{2\pi} \int \exp\left(\frac{-(x - y)^2}{2h}\right) u(x, t) \left[\frac{(x - y)^2}{2h^3} - \frac{1}{h^2}\right] dx.$$

Now, with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, differentiating with respect to y_i gives

$$\frac{\partial u}{\partial y_i} = \frac{1}{2\pi h} \int \exp\left(\frac{-(x - y)^2}{2h}\right) u(x, t) \left[\frac{x_i - y_i}{h}\right] dx,$$
$$\frac{\partial^2 u}{\partial y_i^2} = \frac{1}{2\pi h} \int \exp\left(\frac{-(x - y)^2}{2h}\right) u(x, t) \left[\frac{(x_i - y_i)^2}{h^2} - \frac{1}{h}\right] dx.$$

Thus

$$\nabla^2 u = \frac{\partial^2 u}{\partial y_1^2} + \frac{\partial^2 u}{\partial y_2^2}$$
$$= \frac{1}{2\pi} \int \exp\left(\frac{-(x-y)^2}{2h}\right) u(x,t) \left[\frac{(x-y)^2}{h^3} - \frac{2}{h^2}\right] dx,$$

so $\dfrac{\partial u}{\partial h} = \dfrac{1}{2}\nabla^2 u.$

18.4 We first establish the identity

$$\int \frac{\nabla^2 \psi(x)}{|y-x|} dx = -4\pi\psi(y)$$

for a smooth (twice continuously differentiable, say) $\psi(x)$ that is zero for all sufficiently large x. To see this, note that for small $\epsilon > 0$, since $\nabla^2(1/1y - x1) = 0$ for $x \neq y$,

$$\int_{|y-x|\geq\epsilon} \frac{\nabla^2\psi(x)dx}{|y-x|} = \int_{|y-x|\geq\epsilon} \left[\frac{\nabla^2\psi(x)}{|y-x|} - \psi(x)\nabla^2 \frac{1}{|y-x|}\right] dx$$
$$= -\int_{|y-x|=\epsilon} \left[\frac{\nabla\psi}{|y-x|} - \psi(x)\nabla\frac{1}{|y-x|}\right] .dn(x) \quad (1)$$

where $n(x)$ denotes the outwards pointing unit normal at x on the sphere $|y-x| = \epsilon$. Here we have used Green's theorem for a region between a sphere of radius ϵ and a large sphere on which $\psi(x) = 0$. Differentiating with respect to each coordinate gives $\nabla_x \dfrac{1}{|y-x|} = -\dfrac{(x-y)}{|y-x|^3}$. Hence (1) gives

$$\int \frac{\nabla^2\psi(x)}{|y-x|} dx = \lim_{\epsilon\to 0} \int_{|y-x|\geq\epsilon} \frac{\nabla^2\psi(x)}{|y-x|} dx$$
$$= -\lim_{\epsilon\to 0} \int_{|y-x|=\epsilon} \left[\frac{\nabla\psi}{\epsilon} + \psi(x)\frac{(x-y)}{\epsilon^3}\right] .dn(x)$$
$$= -4\pi\psi(y), \quad (2)$$

since ϕ is continuous at y, and $\int_{|y-x|=\epsilon} \nabla\psi\epsilon^{-1}.\, dn(x) = O(\epsilon)$, and moreover $\int_{|y-x|=\epsilon} (x-y).dn(x) = 4\pi\epsilon\epsilon^2$.

Now let f be continuous and integrable on $\mathbb{R}^3$ and let $\phi(x) = \displaystyle\int \frac{f(y)}{|y-x|} dy$. Let $\psi(x)$ be any smooth function vanishing for all sufficiently large x. Using

Green's formula,

$$\begin{aligned}\int \psi \nabla^2 \phi dx &= \int \phi \nabla^2 \psi dx = \iint \frac{f(y)\nabla^2\psi(x)}{|y-x|} dx\, dy \\ &= \int f(y) \int \frac{\nabla^2\psi(x)}{|y-x|} dx\, dy \\ &= -\int 4\pi f(y)\psi(y)\end{aligned}$$

using (2). Hence, $\int(\nabla^2\phi + 4\pi f)\psi = 0$ for all smooth ψ vanishing for large x, so by an orthogonality argument, $\nabla^2\phi + 4\pi f = 0$, as required.

18.5 Assume that $f(x) = 0$ for $x \notin B(0, r)$. Using the Cauchy-Schwartz inequality and setting $R = r + |x|$, we have

$$\begin{aligned}\phi(x)^2 &= \left(\int_{B(0,r)} \frac{f(y)dy}{|x-y|}\right)^2 \\ &\le \int_{B(0,r)} f(y)^2 dy \int_{B(0,r)} \frac{dy}{|x-y|^2} \\ &\le c \int_{B(0,R)} \frac{dy}{|x-y|^2} \\ &= c \int_{\theta \in S} \int_{r=0}^{R} \frac{r^2 dr d\theta}{r^2} < \infty,\end{aligned}$$

changing to polar coordinates, with S the unit sphere. Thus the singularity set of x, that is x such that $\phi(x) = \infty$, is empty.

18.6 Let $\mu = \nu \times m$ be the measure on $F = D \times L$ where ν is t-dimensional Hausdorff measure restricted to a self-similar Cantor dust of dimension t (satisfying the open set condition) and m is the restriction of 1-dimensional Lebesgue measure to the line segment L. Then, as in Exercise 9.11, there are constants c_1, c_2 such that if $x \in D$ and $r < 1/2$, we have $c_1 r^t \le \nu(B(x, r)) \le c_2 r^t$. Writing $C(w, r)$ for the cylinder with axis $L(w)$ parallel to L, center $w = (x, z) \in D \times L$, radius r and height $2r$, it follows that

$$\begin{aligned}c_1 r^{t+1} &= c_1 r^t r \le \nu(B(x, r)) \times m(L(w)) \\ &= \mu(C(w, r)) \le c_2 r^t 2r = 2c_2 r^{t+1}.\end{aligned}$$

Since $C(w, 2^{-1/2} r) \subset B(w, r) \subset C(w, r)$, it follows that

$$2^{-(t+1)/2} c_1 r^{t+1} \le \mu(B(w, r)) \le 2c_2 r^{t+1}.$$

With $s = t + 1$ and redefining c_1 and c_2, we are in the situation of Section 18.3, again with

$$\int_{|h|\le r} \langle \epsilon(x)\epsilon(x+h)dh\rangle \sim r^s,$$

but this time with the dissipation occurs around the stratified set F.

18.7 For a von Koch curve of base length a, the resonant wavelengths are given by the basic similarity sizes: $a, 3^{-1/2}a, 3^{-1}a, 3^{-3/2}\ldots$ (remembering that the von Koch curve comprises two similar copies at scale $3^{-1/2}$). Thus the resonant frequencies are proportional to the reciprocals $\omega, 3^{1/2}\omega, 3\omega, 3^{3/2}, \ldots$.

18.8 With B^α index-α fractional Brownian motion, we have from (16.10)

$$p(r) \equiv \mathsf{P}(0 \le B^\alpha(t+h) - B^\alpha(t) \le r) = \frac{1}{\sqrt{2\pi}h^\alpha}\int_0^r \exp\left(\frac{-x^2}{2h^{2\alpha}}\right)dx,$$

In particular

$$\begin{aligned}\mathsf{E}(|B^\alpha(t+h) - B^\alpha(t)|^q) &= 2\int_0^\infty r^q dp(r)\\ &= \frac{2}{\sqrt{2\pi}h^\alpha}\int_0^\infty r^q \exp\left(\frac{-r^2}{2h^{2\alpha}}\right)dr\\ &= ch^{q\alpha}\int_0^\infty u^q \exp\left(\frac{-u^2}{2}\right)du\end{aligned}$$

on setting $u = r/h^\alpha$. Thus

$$\begin{aligned}\mathsf{E}(|X(t+h) - X(t)|^q) &= \mathsf{E}(|B^\alpha(T(t+h)) - B^\alpha(T(t))|^q)\\ &= c_1|T(t+h) - T(t)|^{q\alpha}\\ &= c_1\mu[t, t+h]^{q\alpha} \sim h^{\gamma q\alpha}\end{aligned}$$

for $t \in E_\gamma$.